REPUBLIC OF TURKEY
YILDIZ TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

SPECIAL FUNCTIONS IN TRANSFERRING OF ENERGY AND SURPLUS OF ENERGY

NEVRA EREN

PH.D. THESIS
DEPARTMENT OF MATHEMATICAL ENGINEERING
PROGRAM OF MATHEMATICAL ENGINEERING

ADVISER
ASSOC. PROF. DR. KEVSER KÖKLÜ

İSTANBUL, 2015

REPUBLIC OF TURKEY

YILDIZ TECHNICAL UNIVERSITY

GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

SPECIAL FUNCTIONS IN TRANSFERRING OF ENERGY AND SURPLUS OF ENERGY

A thesis submitted by Nevra EREN in partial fulfillment of the requirements for the degree of **DOCTOR OF PHILOSOPHY** is approved by the committee on 10.07.2015 in Department of Mathematical Engineering, Mathematical Engineering Program.

Thesis Adviser

Assoc. Prof. Dr. Kevser KÖKLÜ

Yıldız Technical University

Co- Adviser

Assist. Prof. Dr. Emre EROĞLU

Kırklareli University

Approved By the Examining Committee

Assoc. Prof. Dr. Kevser KÖKLÜ

Yıldız Technical University

Assist. Prof. Dr. Emre EROĞLU, Member

Kırklareli University

Assist. Prof. Dr.Nihan ÇINAR, Member

Kırklareli University

Prof. Dr İsmail KÖMBE, Member

Istanbul Commerce University

Prof. Dr. İbrahim GÜNEY, Member

Yıldız Technical University

Assist. Prof. Dr.Hasan Hüseyin YURTÇU, Member

Yıldız Technical University

Assoc. Prof. Dr.Ali SIRMA, Member

Istanbul Technical University

This study was supported by Kırklareli University, **KUBAP 09.**

ACKNOWLEDGEMENTS

I would like to take this opportunity to thank all of those who have helped me to prepare my Ph.D. thesis at Yıldız Technical University and Kırklareli University. I am deeply in depted to my advisers, Doç. Dr. Kevser Köklü and Yrd. Doç. Dr. Emre Eroğlu. They have taught me how to prepare this thesis in a more effective way. I would like to express my appreciations to my thesis committee members; Doç. Dr. Kevser Köklü, Yrd. Doç. Dr. Nihan Çınar and my second adviser Yrd. Doç. Dr. Emre Eroğlu for the time and consideration given to this study.

I would like to express my special thanks to Doç. Dr. Kevser Köklü, Yrd. Doç. Dr. Emre Eroğlu and some of my friends who never let me alone, for all their supportive manners and advices.

This thesis was supported by Kırklareli University, **KUBAP 09**.

Finally, I wish to thank my mother, my father, my elder-sister and all my children. I dedicate this thesis to those who always belive me and never give up supporting me.

July, 2015

Nevra EREN

TABLE OF CONTENTS

Page

LIST OF SYMBOLS

$A_i(.)$	Airy function of the first kind
$B_i(.)$	Airy function of the second kind
c	Velocity of light in vacuum (m/s)
∂_t	Partial derivative with respect to t (1/s)
∂_n	Normal derivative
∂_z	Partial derivative with respect to z (1/m)
E_m	Electric field vector (V/m)
E(r)	Transverse of electric field vector with 3-component
$e_n(z,t)$	Potential function of KGE for the TM modes
ε_0	Dielectric constant of free space (F/m)
F	Excitation function in waveguide form
G , J	Operators decomposed from Maxwell's Equations
H_m	Magnetic field vector (A/m)
$H(\mathrm{r})$	Transverse of magnetic field vector with 3-component
$h_n(z,t)$	Potential function of KGE for the TE modes
κ_n	Eigenvalues of Dricihlet problem
κ_n^2	Eigenvalues to the Dirichlet boundary-eigenvalue problem $(1/\mathrm{m}^2)$
L	Contour of the waveguide cross section
S	Waveguide cross section
r	Projection of the position vector R onto the domain S (m)
t	Observation Time (s)
X_m	$col(\mathrm{E}_m(\mathrm{r,t}), \mathrm{H}_m(\mathrm{r,t}))$ columun vector
$X(\mathrm{r})$	col(E(r) , H(r)) columun vector with 6-component
υ_n^2	Eigenvalues to the Neumann boundary-eigenvalue problem $(1/\mathrm{m}^2)$
μ_0	Magnetic permeability of free space (H/m)
∇	Nabla operator
$\nabla_\perp$	Nabla transverse
$\nabla^2{}_\perp$	Transverse Laplacian
α	Constant of seperation of variables
τ	Dimensionless time
ξ	Dimensionless coordinate
$\bar{\gamma}$	Lorentz factor
ψ_n	Eigenfunctions to the Neumann boundary-eigenvalue problem
ϕ_n	Eigenfunctions to the Dirichlet boundary-eigenvalue problem

LIST OF ABBREVIATIONS

A_i	Airy Function of the First Kind
B_i	Airy Function of the Second Kind
EAE	Evolutionary Approach to Electromagnetics
KGE	Klein-Gordon Equation
TE	Transverse Electric
TM	Transverse Magnetic

LIST OF FIGURES

ABSTRACT

SPECIAL FUNCTIONS IN TRANSFERRING OF ENERGY AND SURPLUS OF ENERGY

Nevra EREN

Department of Mathematical Engineering

Ph.D. Thesis

Advisor: Assoc. Prof. Dr. Kevser KÖKLÜ

Co-advisor: Assist. Prof. Dr. Emre EROĞLU

In this study,the electromagnetic fields created by an excited source function and surplus of energy of signal propagation for time-domain modes have been discussed with an analitical method called the Evolutionary Approach to Electromagnetics (EAE) in a hollow waveguide. The hollow waveguide means that there are no charges and currents.The surface of the waveguide is perfectly conducting. The problem has been solved in a cross-section that is singly connected but geometrically surrounded by a regular contour L along oz-axis. The subject of this study is real valued electromagnetic fields that are defined as E(R,t) and H(R,t). They are electric and magnetic field force vectors, respectively. Analysis of transferring of transient signals along time-harmonic waveguide modes consists of two main parts. The first one is a modal basis problem. The second one is a modal amplitude problem. Surplus of energy is observed in process of the signal transferring. Surplus of energy for the time-domain waveguide modes are presented via Airy function.

Firstly, Maxwell’s equations with time derivative have been formulated and some properties of them have been given. The differential form has been obtained for the energy conservation with the help of Poynting’s theorem. The standart Maxwell’s equations system has been rearranged in transverse and longitudinal coordinates. Each electric and magnetic field components is a product of modal base and modal amplitude. In the time –space, the modal bases have been obtained by solving Neumann and Dirichlet boundary eigenvalue problems. Then, the scalar functions which are called modal amplitudes have been found with the help of Klein-Gordon Equation (KGE). It has been shown that Klein-Gordon Equation is invariant under the relativistic Lorentz transformations. Finally, the solutions to KGE, modal amplitudes, have been formulated as product of Airy functions and the energetic properties of the modal fields have been investigated. Some numerical examples for energy , surplus of energy have been given with the help of graphics.

Key words: Maxwell's equations, Klein-Gordon equation, time-domain modes, Airy functions, surplus of energy, evolutionary equations, the Evolutionary Approach to Electromagnetics.

ÖZET

ENERJİ TRANSFERİNDE ÖZEL FONKSİYONLAR VE ENERJİ DEĞİŞİMİ

Nevra EREN

Matematik Mühendisliği Anabilim Dalı

Doktora Tezi

Tez Danışmanı: Doç. Dr. Kevser KÖKLÜ

Eş Danışman: Yrd. Doç. Dr. Emre EROĞLU

Bu çalışmada, uyarıcı bir kaynak fonksiyonu tarafından oluşturulan elektromanyetik alanlar ve zaman-uzayı modları için boş bir dalga kılavuzunda sinyal yayılımının oluşturduğu enerji değişimi (artımı) analitik bir metod olan Elektromanyetik Teoriye Evrimsel Yaklaşım (ETEY) metodu ile ele alınmıştır. Boş bir dalga kılavuzu yük ve akımın olmadığını ifade eder. Dalga kılavuzu yüzeyi mükemmel bir elektrik iletkendir. OZ-ekseni boyunca düzgün bir L kontürü ile çevrili, geometrik olarak tekil bağlantılı bir kesit-alanda problem çözülmüştür. Çalışma konusu reel değerli elektromagnetik alanlardır. Bunlar sırasıyla $\mathrm{E}(\mathrm{R},\mathrm{t})$ ve $\mathrm{H}(\mathrm{R},\mathrm{t})$, elektrik ve magnetik alan kuvvet vektörleridir. Geçici sinyallerin zaman-harmonik dalga kılavuzu modları boyunca transferinin analizi iki ana bölümden oluşur. Birincisi modal baz problemidir. İkincisi modal genlik problemidir. Sinyal transferi sürecinde enerji değişimi gözlemlenir. Burada zaman-uzayı dalga kılavuzu modlarının enerji değişimi (artımı) Airy fonksiyonları yoluyla sunulmuştur.

İlk olarak, zaman türevli Maxwell denklemleri formülize edilmiştir ve bazı özellikleri verilmiştir. Poynting Teoremi yardımıyla enerji korunumu için diferansiyel form elde edilmiştir. Standart Maxwell denklemler sistemi enlemsel ve eksenel koordinatlarda yeniden düzenlenmiştir. Her bir elektrik ve manyetik alan bileşenleri modal baz ve modal genlik çarpımıdır. Zaman-uzayında , modal bazlar Neumann ve Dirichlet sınır özdeğer problemlerinin çözülmesiyle elde edilmiştir. Sonra, Klein-Gordon denklemi yardımıyla modal genlik denilen skaler fonksiyonlar elde edilmiştir. Klein-Gordon denkleminin relativistik Lorentz dönüşümleri altında değişmez (invariant) olduğu gösterilmiştir.Son olarak, Klein-Gordon denkleminin çözümleri Airy fonksiyonları çarpımı olarak formülize edilmiştir ve modal alanların enerjik özellikleri araştırılmıştır. Enerji ve enerji artımı için bazı sayısal örnekler grafikler yardımıyla verilmiştir.

Anahtar Kelimeler: Maxwell denklemleri, Klein-Gordon denklemi, zaman-uzayı modları, Airy fonksiyonları, enerji değişimi, evrimsel denklemler, Elektromanyetik Teoriye Evrimsel Yaklaşım.

CHAPTER 1

INTRODUCTION

Investigation of electromagnetic fields at finite boundary-value problems is a main problem in electromagnetic theory. Structures of the problem have been solved are called cavity or resonator. It is possible to discuss transferring from $t = -\infty$ to $t = +\infty$ in classical time-harmonic electromagnetic theory, theoretically. But in practice electromagnetic fields need to source for excitation. Electromagnetic fields are just oscillated from beginning such as $t = 0$ [1].

Problems encountered in engineering theory and practice in the areas of energy essential to use mathematical equations [2] and [3]. Time-harmonic studies are accumulated Finite Difference Time Domain (FTDT) method extensively [4]. This method is numerical. On the other hand, analytical methods are used as well. Fourier and Laplace integral transformations are an analytical method. The remarkable approaches for time-domain solutions were studied in electrodynamics [5].

Mathematical results must satisfy physical requirements in the analytical method. Evolutionary Approach To Electromagnetic Theory In Time Domain (EAE) is also an analytical method [6]. The method investigates solutions of elctromagnetic fields in time-domain analytically. Papers have demonstrated with the EAE method transferring of signal on perfect electric conductor surfaces and their cross-section recently [7], [8].

Analysis of transferring of transient signals along time-harmonic waveguide modes consists of two main parts. The first one is presentation of a modal basis problem. The second one is a modal amplitude problem that clarity gained by solving Klein-Gordon equation. Solution of Klein-Gordon equation has revealed time-harmonic modal amplitudes. Introduction of Klein-Gordon equation is founded by obeying causality principle. The solution of Klein-Gordon equation is represented serially. Surplus of energy could be observed in process of the signal transferring. In this study, it is demonstrated the energetic considerations for Airy function [9]. Surplusof energy for the waveguide time-domain waveguide modes are represented via Airy function. The previous works are reviewed for energy and surplus of energy of time-domain modes in the waveguides [10].

1.1 Literature Review

The discussions in this dissertation has been carried out in the framework of the Evolutionary Approach to Electromagnetics (EAE) method. The main principle of EAE to find electromagnetic fields is to preserve time as an independent variable in the process of solving Maxwell's equations with time derivative, ∂_t. Then, the amplitudes and potentials have been derived from Klein-Gordon and Helmholtz equations, respectively. After accepting that the electromagnetic theory is establihed on Maxwell's equations, most of the problems are developped in frequency (Fourier) space. The use of Fourier transform in solving electromagnetic problems eliminates the time as an independent variable. The solutions to electromagnetic problems in the time domain are investigated by applying inverse Fourier transforms. Thus, the problems are described as obtaining the effective frequencies to calculate the amplitudes of the fields at these frequencies [1].

In contrast to the use of Fourier transforms, the method of EAE provides us to calculate and observe the differential equations with time derivative, ∂_t in the interval $t_0 = 0$ and $t_1 = t$. The differential equations with time derivatives are named as evolution equations by mathematicians. The main point of the method is to observe and calculate the electromagnetic fields in the time space under the suitable initial and boundary conditions. These boundary conditions are dependent on the cross sectional area on which the problem is studied. So, this method is called Evolutionary Approach. The modal amplitudes derived from the KGE are functions of axial coordinate z and time. The solution obtained from the equation is also a serial solution.

The method was introduced in the 1980's and early studies were published in Russian scientific papers [15] and [16]. The English version of the papers were published in 1990's [6], [17]. For different applications of the EAE method, the recent publications can be used to get an idea [7], [11], [12].

The EAE method is based on two main ideas. A chronological examination of the publishes gives us that the pioneers of the idea of working electromagnetic problems in the time space seen in 1940's and 1950's. J.C.Slater [18], G.V.Kisun'ko, K.Kurokawa [19] and R.Müller are the first people that they tried to introduce the fields as series of eigenmodes. Later, in electromagnetic theory this idea was explained in detail by J.Van Bladel [20]. This idea has been proposed for the complex amplitudes of the fields and it has continued to be implemented up to now. By applying the Fourier transform to Maxwell's equations in the time space, ''∂_t '' is replaced by '' iw '' and the non-linear constitutive equations are made linear. Hence, only one thing remains: linearization of the electromagnetic problem in frecuency domain. But even in the linear electromagnetic problems Fourier transform is not as easy as it looks [1]. An in-depth review of the problems encountered in this regard is available in P.Hillion's article [21].

The second important idea is to obtain the eigenmod solutions conserving ∂_t in Maxwell's equations. The modal amplitudes which are the coefficients in these series should be time dependent. As a result, evolutionary differential equations with time derivative are obtained for solving modal amplitudes. These equations open another way for the development of the electromagnetic theory in the time space; The Evolutionary Approach for Electromagnetic Theory (EAE). Actually, the constitutive equations have the differential form that contains time derivative. To be converted to their algebraic form, force term starts by solving the motion equation which is regarded as a harmonic mark H.F.Harmuth has reminded of this fact in his recent publications [22]. The differential form of the constitutive equations with time derivative is natural for the EAE method, so it is easy to combine the constitutive equations with the evolution equation [1]. The EAE method can be carried out in different versions. Each of these starts with the mathematical decompositions of self-adjoint or only symmetric linear operator from Maxwell's equations. The second mentioned space acts in accordance with its variables. The decomposition of the self-adjoint operator from Maxwell's equation in this way gives an operator eigenvalue equation. The egienvalues of these egienequations form a basis of the selected solution space. As a result, the solution which is tried to find may be presented in terms of eigenvectors series. Physically, this gives the modal field solutions and where the modal amplitudes are time- dependent. Then, the evolution equations for the modal amplitudes are derived [6], [1].

The EAE method has practical examples to cavity problems [6]. First, a self-adjoint operator which is applied to inside an empty cavity has been decomposed from

Maxwell's equations. Then, this cavity can be filled with the desired material. Dispersive or lossy material can be used here [1].

Electromagnetic fields in a waveguide have been obtained in terms of the extension of the modal bases elements. These elements of bases are vector functions of the coordinates. Every element in the series has a scalar modal amplitude factor. These amplitudes are required functions for the fields that can be written in terms of time dependency. In general, base elements have been expressed with the vector boundary eigenvalue problem for the Laplace operator. In the special cases of all that the separation of variables can be applied to, these vector problems have been transformed into Dirichlet and Neumann boundary value problems for the Laplace operator. An ordinary differential equations system with time derivative has been derived from Maxwell's equations for the modal amplitudes. The system has supported by suitable initial conditions. This problem is a Cauchy problem. Such modal amplitudes have been calculated as simple convolution integrals. The source sign may be an arbitrary function of the time. It has been guaranteed to satisfy the causality principle [1].

1.2 Objective of the Thesis

The analytical method that is used depends on the time differential Maxwell's equations which can be formulated in transverse and longitudinal coordinates. These equations have been solved analytically. A radiating electromagnetic wave in space also carries energy with it. Therefore, the energetic properties of the electric and magnetic field have been examined. The waveforms of the energy exchange among certain components of the modal fields must also act in conjunction with radiation of electric-magnetic field due to the energy of the time- dependent. In the study the scalar functions providing the signal transfer have been shown graphically with the energy they carry. These scalar functions are solutions to the KGE.

General framework of this study has been published in the article that titled as '' Special functions in transferring of energy; a specific case: Airy function''. So, it is based on reference [13].This study has been done step by step as follow. A hollow (i.e., medium-free) waveguide with its cross-section domain *S* bounded by a closed singly connected contour *L* is considered. It is supposed that *L* has enough smooth shape which implies that none of possible inner angles of *L* (i.e., being measured within *S*) exceeds π and the cross section *S* maintains its form and size along the waveguide axis o*z*. Our aim is to solve the modal fields for the *TE* and *TM* modes which are a particular solution to the system of Maxwell's equations with the time derivative given as

$$\nabla x\boldsymbol{E}(\boldsymbol{R},\boldsymbol{t}) = -\mu_0 \frac{\partial}{\partial t}\boldsymbol{H}(\boldsymbol{R},t) \ , \qquad \nabla x\boldsymbol{H}(\boldsymbol{R},t) = \varepsilon_0 \frac{\partial}{\partial t}\boldsymbol{E}(\boldsymbol{R},t) \tag{1.1}$$

where $\mathbf{E}(\mathbf{R},t)$ and $\mathbf{H}(\mathbf{R},t)$ are the electric and magnetic fields, respectively. ε_0 and μ_0 are dielectric and magnetic constants for free-space, respectively. Because the fields will be excited by an initial condition technique, the source term is not considered in the Maxwell's equations. The vector $\mathbf{R}$ within the waveguide volume denotes an observation point. t is observation time. Let's introduce a right-handed triplet of the mutually orthogonal unit vectors $(\mathbf{z},\boldsymbol{l},\boldsymbol{n})$ where $\mathbf{z}\times\boldsymbol{l}=\boldsymbol{n}$. The unit vector $\mathbf{z}$ and $\boldsymbol{l}$ are tangential to the axis o*z* and contour *L*, respectively. The unit vector $\boldsymbol{n}$ is outward normal to the cross-section of domain *S* [8].

Let's decompose the vector $\mathbf{R}$ and Nabla operator ∇ onto their transverse and longitudinal parts as

$$\boldsymbol{R} = \boldsymbol{r} + \boldsymbol{z}z, \qquad \nabla = \nabla_{\perp} + \mathbf{z}\partial_z \tag{1.2}$$

where the projection $\mathbf{r}$ is a position vector within the domain *S* and $\nabla_{\perp}$ is the transverse Laplacian operator.

Subject of our study is real-valued electromagnetic fields specified by the electric and magnetic field strength vectors $\mathbf{E}_m(\mathbf{R},t)$ and $\mathbf{H}_m(\mathbf{R},t)$, respectively. Separate these vectors onto their transverse and longitudinal parts similarly to performed in Eq. (1.2), i.e.,

$$\boldsymbol{E_m}(\boldsymbol{R},t) = \boldsymbol{E}(\boldsymbol{r},z,t) + \boldsymbol{z}E_z(\boldsymbol{r},z,t)$$

$$\boldsymbol{H_m}(\boldsymbol{R},t) = \boldsymbol{H}(\boldsymbol{r},z,t) + \boldsymbol{z}H_z(\boldsymbol{r},z,t) \tag{1.3}$$

where $m = 1,2,\ldots$. Because the waveguide surface is supposed to have physical properties of the perfect electric conductor, the following boundary conditions hold over the waveguide surface

$$\boldsymbol{n}.\boldsymbol{H_m}(\boldsymbol{R},t)|_L = 0 \ , \quad \boldsymbol{l}.\boldsymbol{E_m}(\boldsymbol{R},t)|_L = 0 \ , \quad \boldsymbol{z}.\boldsymbol{E_m}(\boldsymbol{R},t)|_L = 0 \tag{1.4}$$

Let's give a summary about the contents of this study in general.

Firstly, the modal basis problem has been studied. Here TE time- domain modes and TM time-domain modes have been investigated.

For *TE* time-domain modes, let's consider the Neumann boundary eigenvalue problem for the operator $\nabla_\perp^2$ as

$$(\nabla_\perp{}^2 + v_m{}^2)\psi_m(\boldsymbol{r}) = 0 \ , \qquad \left.\frac{\partial\psi_m(\boldsymbol{r})}{\partial n}\right|_L = 0 \ , \quad \frac{\mu_0 v_m^2}{S}\int_S |\psi_m(\boldsymbol{r})|^2\, ds = 1\, N \quad (1.5)$$

where $\partial_n = \mathbf{n}\cdot\nabla_\perp$ is the normal derivative on the contour *L*. $\upsilon_m^2 > 0$ and $m = 1,2,3,..$ are the eigenvalues and their regulation numbers of position on a real axis in the increasing order of their numerical values. The potentials $\psi_m(\mathbf{r})$ are the eigenvectors of the corresponding eigenvalues.

For eigenvalue $\upsilon_0^2 = 0$, the problem (1.5) will have the following form

$$\nabla_\perp{}^2\psi_0(\boldsymbol{r}) = 0\,, \quad \left.\frac{\partial\psi_m(\boldsymbol{r})}{\partial n}\right|_L = 0 \qquad (1.6)$$

where the function $\psi_0(\mathbf{r})$ is a harmonic function and its value is distinct from zero. The minimum-maximum theorem for the harmonic functions yields that $\psi_0(\mathbf{r}) = C$ where $r \in L + S$ and *C* is an arbitrary constant [10].

Every particular solution $\psi_m(\mathbf{r})$ to the Neumann problem (1.5) generates the *TE* time-domain modal fields with the components as

$$\mathbf{E}_{zm}^h = 0$$

$$v_m^{-1}\mathbf{E}_m^h = \left\langle -\partial_{v_m ct} h_m(z,t)\right\rangle \left[\varepsilon_0^{-\frac{1}{2}}\mu_0^{\frac{1}{2}} v_m A_m^{TE}\nabla_\perp\psi_m(\boldsymbol{r}) x \boldsymbol{z}\right]$$

$$v_m^{-1}\mathbf{H}_m^h = \left\langle \partial_{v_m z} h_m(z,t)\right\rangle [v_m A_m^{TE}\nabla_\perp\psi_m(\boldsymbol{r})] \qquad (1.7)$$

$$v_m^{-1}\mathbf{H}_{zm}^h = \langle h_m(z,t)\rangle [v_m{}^2 A_m^{TE}\psi_m(\boldsymbol{r})]$$

where $\partial_{\upsilon_m ct} = (1/c\upsilon_m)\partial/\partial t,\ \partial_{\upsilon_m z} = (1/\upsilon_m)\partial/\partial z$ and $c = 1/\sqrt{\varepsilon_0\mu_0}$. Specially, the potential $\psi_0(\mathbf{r})$ generates a one-component modal field as

$$\boldsymbol{E}_0(\boldsymbol{r},z,t) = 0 \ , \ \boldsymbol{H}_0(\boldsymbol{r},z,t) = \boldsymbol{z}C\,. \qquad (1.8)$$

The potential $h_m(z,t)$ in Eq. (1.7) is governed by Klein-Gordon Equation (KGE)

$$(\partial^2{}_{v_m ct} - \partial^2{}_{v_m z} + 1)\, h_m(z,t) = 0 \tag{1.9}$$

which is known as a generalized wave equation [9] and [11].

As similar to the problem of the TE time-domain modes, the Dirichlet boundary eigenvalue problem for the operator $\nabla_\perp^2$ can be stated as follows

$$(\nabla_\perp{}^2 + \kappa_m{}^2)\phi_m(\boldsymbol{r}) = 0\ , \qquad \phi_m(\boldsymbol{r})|_L = 0\ , \quad \frac{\varepsilon_0 \kappa_m^2}{S}\int_S |\phi_m(\boldsymbol{r})|^2\, ds = 1\, N \tag{1.10}$$

where $\kappa_m{}^2 > 0, m = 1,2,3..$ are the eigenvalues. The potential $\phi_0(\mathbf{r})$ will be zero.

The solution $\phi_m(\mathbf{r})$ to the Dirichlet problem (1.10) generates the TM time-domain modal fields with the following components

$$\mathbf{H}_{zm}^e = 0$$

$$\kappa_m^{-1}\mathbf{H}_m^e = \left\langle -\partial_{\kappa_m ct} e_m(z,t)\right\rangle \left[\boldsymbol{z} x \mu_0^{-\frac{1}{2}} \varepsilon_0^{\frac{1}{2}} \kappa_m A_m^{TM} \nabla_\perp \phi_m(\boldsymbol{r})\right]$$

$$\kappa_m^{-1}\mathbf{E}_m^e = \left\langle \partial_{\kappa_m z} e_m(z,t)\right\rangle [\kappa_m A_m^{TM} \nabla_\perp \phi_m(\boldsymbol{r})] \tag{1.11}$$

$$\kappa_m^{-1}\mathbf{E}_{zm}^e = \langle e_m(z,t)\rangle [\kappa_m{}^2 A_m^{TM} \phi_m(\boldsymbol{r})]$$

where $\partial_{\kappa_m ct} = (1/c\kappa_m)\partial/\partial t,\ \partial_{\kappa_m z} = (1/\kappa_m)\partial/\partial z$. The potential $e_m(z,t)$ generates the modal amplitudes in Eq. (1.11) is the solution of the KGE as

$$(\partial^2{}_{\kappa_m ct} - \partial^2{}_{\kappa_m z} + 1)\, e_m(z,t) = 0 \tag{1.12}$$

which is similar to Eq. (1.9),[10], [11].

The set of the TE and TM modes (as the vector functions of transverse coordinates) is complete due to the completeness of their generating potentials in the same energetic space. The completeness comes from Sturm-Liouville and Weyl theorem in functional analysis about the orthogonal detachments of Hilbert space $L_2(S)$ [6], [11], [47]. This energetic space can be specified by an inner product as

$$(X_1, X_2) = \frac{1}{S}\int_S (\varepsilon_0 \boldsymbol{E}_1 . \boldsymbol{E}_2 + \boldsymbol{\mu_0}\, \boldsymbol{H}_1 . \boldsymbol{H}_2) ds < \infty \tag{1.13}$$

where $X_i = col(\mathbf{E}_i, \mathbf{H}_i),\ i = 1,2,..,$ col. stands for ‘‘column”. One can verify that $(X_m^{TE}, X_n^{TM}) = 0$ for any combinations of *m* and *n* with the values $0,1,2,...$ independently. Therefore, any pair of the TE and TM time-domain modes is orthogonal in the sense of inner product (1.13), [10].

Secondly, the time-dependent modal amplitude problem has been studied.

The KGE in Eq. (1.9) for the TE modes and the KGE in Eq. (1.12) for the TM modes have the same structure. After introducing the scaled time τ and scaled coordinate ξ as

$$\tau = \nu_m ct \quad , \quad \xi = \nu_m z \qquad for\ TE - modes$$

$$\tau = \kappa_m ct \quad , \quad \xi = \kappa_m z \qquad for\ TM - modes \tag{1.14}$$

the KGE in Eq. (1.9) and Eq. (1.12) can be written in the general form of

$$(\partial^2{}_\tau - \partial^2{}_\xi + 1)\, f(\xi,\tau) = 0 \tag{1.15}$$

where $f(\xi,\tau)$ is either $e_m(\xi,\tau)$ provided that $\xi = \kappa_m z$ and $\tau = \kappa_m ct$ or $h_m(\xi,\tau)$ provided that $\xi = \upsilon_m z$ and $\tau = \upsilon_m ct$.

The KGE maintains its form under an action of a Poincare group within the framework of the group theory. In this aspect, Miller established eleven so called orbits of symmetry in terms of the group theory [14]. His results are crucial for development of the electromagnetic field theory in the time-domain.

On the basing of Miller’s idea, let us interpret solution to the KGE in Eq. (1.15) as a function with a new arguments, namely: $f \equiv f(\xi,\tau) = f[u(\xi,\tau), v(\xi,\tau)]$. The “new” variables (u,v) are unknown yet, but suppose that they are twice differentiable functions of the “old” variables (ξ,τ). Substitution of $f[u(\xi,\tau), v(\xi,\tau)]$ as a formal solution to Eq. (1.15) yields a new form of this equation as

$$\left[\left(\frac{\partial u}{\partial \tau}\right)^2 - \left(\frac{\partial u}{\partial \xi}\right)^2\right]\frac{\partial^2 f}{\partial u^2} + \left[\left(\frac{\partial v}{\partial \tau}\right)^2 - \left(\frac{\partial v}{\partial \xi}\right)^2\right]\frac{\partial^2 f}{\partial v^2} + \left[\frac{\partial^2 u}{\partial \tau^2} - \frac{\partial^2 u}{\partial \xi^2}\right]\frac{\partial f}{\partial u}$$

$$+ \left[\frac{\partial^2 v}{\partial \tau^2} - \frac{\partial^2 v}{\partial \xi^2}\right]\frac{\partial f}{\partial v} + 2\left[\frac{\partial u}{\partial \tau}\frac{\partial u}{\partial \tau} - \frac{\partial u}{\partial \xi}\frac{\partial v}{\partial \xi}\right]\frac{\partial^2 f}{\partial u \partial v} + f = 0 \tag{1.16}$$

where notice that the derivatives ∂_u and ∂_v act on the function $f(u,v)$ under study.

Finallly, Surplus of energy for the time-domain mode has been studied.

We think over surplus of energy via Airy function which finds its place in the case 5, from the Miller's eleven cases. In the case 5,

$$2(u+v)=\tau+\xi \quad and \quad (u-v)^2=\tau-\xi \tag{1.17}$$

given the $(-)$ rather than $(\pm)$,

$$u=\frac{(\tau+\xi\ )}{4}-\frac{\sqrt{\tau-\xi}}{2} \quad and \quad v=\frac{(\tau+\xi\ )}{4}+\frac{\sqrt{\tau-\xi}}{2} \tag{1.18}$$

are reversed formulas [9]. Institutions are substituted in equation (1.16), and it is obtained that

$$\left[\left(\frac{\partial u}{\partial \tau}\right)^2-\left(\frac{\partial u}{\partial \xi}\right)^2\right]=-\left[\left(\frac{\partial v}{\partial \tau}\right)^2-\left(\frac{\partial v}{\partial \xi}\right)^2\right]=\frac{1}{4(u-v)} \tag{1.19}$$

and by making (1.16) more useful

$$\frac{\partial^2}{\partial u^2}f(u,v)+4uf(u,v)=\frac{\partial^2}{\partial v^2}f(u,v)+4vf(u,v) \tag{1.20}$$

is obtained.

If $f(u,v)$ is taken as $f(u,v)=U(u)V(v)$,the equation (1.20)turns into the following equation.

$$\frac{1}{U(u)}\frac{d^2}{du^2}U(u)+4u=\frac{1}{V(v)}\frac{d^2}{dv^2}V(v)+4v=4\alpha \tag{1.21}$$

where α is a constant of separation of variables. Final solution to KGE which in compliance with causality principle is introduced as

$$f(\xi,\tau)=\begin{cases} 0 & if\ \tau<0 \\ U(u)V(v) & if\ 0<\xi<\tau \\ 0 & if\ \xi>\tau \end{cases} \tag{1.22}$$

In this case, surplus of energy characteristics for all the time-domain modes can be represented as

$$sW(\xi,\tau)=\frac{[\mathcal{A}^2(\xi,\tau)-\mathfrak{B}^2(\xi,\tau)]}{2} \tag{1.23}$$

Where $f(\xi, \tau)$ represents the solution of the KGE and

$$\mathcal{A} = -\frac{\partial}{\partial \tau} f(\xi, \tau) \qquad and \qquad \mathfrak{B} = \frac{\partial}{\partial \xi} f(\xi, \tau)$$

1.3 Hypothesis

In electromagnetic theory, there are three types of canonic initial-boundary value problems to find electromagnetic field forces by using evolutionary approach, with time derivative. These problems are the cavite problem, the waveguide problem and the outer problem. In this doctorial dissertation, the waveguide problem in a subspace of Euclidean space has been studied. The problem is canonic. It has been solved by using a method which has been used in cavite problem. This method is the decomposition of the Maxwell operator from the equation, in a hollow singly-connected waveguide with perfectly conducting surface. The cross section of the waveguide is regular along the oz axis .

In electromagnetic theory, to study electromagnetic fields by solving an initial-boundary value problem is a basic problem. The structures in which the problems are solved are called ‘‘cavite’’ or ‘‘resonator’’ . In classic time-harmonic electromagnetic theory, it is only possible to study the events start at t=0 continue up to t=∞. But in practic, researchers use oscillations which are called ‘‘forced oscillations’’ to do their procedure. The word ‘‘forced’’ means that the the electromagnetic field occurs when a stimulating source is applied. Electromagnetic fields occur and start to oscillate after applying the source. Of course, there is a time t=0 for such an event [1].

It is familiar to do time domain studies by two known methods. The most used one is ‘‘Finite Difference Time Domain –FDTD’’ method. This method is possible by application of numeric analysis. By this method numeric quantities can be analized with numbers [4]. During oscillations the data can be obtained and these data can be recorded, then compared to each other. So, one can observe the procedure in every step. This method provides us to compare the data obtained from computer [1].

The second method is analytical. In this method, Fourier and Laplace integral transformations are used. The causality principle is one of the principles should be underlined because the integrals are studied in $(-\infty, \infty)$ interval. Physics and Mathematics should be considered together. Mathematical results must satisfy the physical requirements.

Evolutionary Approach to Electromagnetic theory (EAE) is an analytical method. That is, the method studies the solutions of electromagnetic fields analytically. In analytical studies of them, the differential evolutionary equations are studied. Because of conservation of time derivative, these differential equations are called evolutionary equations. In analytical examination, it is studied with differential evolution equations. Stimulation of electromagnetic fields and transferring of them along waveguide are investigated with a sinusoidal sign which has an initial time. These studies have been made recently with the EAE method [7], [11], [12].

In this study, the research has been carried out in the framework of the Evolutionary Approach to Electromagnetics (EAE) proposed for the study of electromagnetic field propagation in a hollow waveguide. The importance of the method is to preserve time as an independent variable in the process of solving Maxwell's equations with time derivative, ∂_t. Problem of surplus of energy has also been studied. The fifth pair of functions in Miller's list has been used for signal distribution problem by Tretyakov and Akgün [9],[44]. The problem has been discussed and concluded. As a result, the function that leads to signal transferring and the energy that is produced by electromagnetic fields have been shown. Then, the surplus of energy has been found and formalized via Airy functions. Before this study, surplus of energy has been formalized via Bessel functions of integer-order and semi-integer order by Eroğlu, Aksoy and Tretyakov [8]. Formulizing surplus of energy via Airy functions has been demonstrated and discussed for the first time in this thesis. In this new discussion, the energy saved by longitudinal and transverse field components has also been formulated via Airy functions in the time domain.

CHAPTER 2

GENERAL INFORMATION

In this section, some preliminary knowledge used in the study will be given. This provides some motivation for our subject with which the remainder of our study is concerned.

2.1 Airy Functions

This section is devoted to general definitions and properties of Airy functions as they can be, at least partially, found in the chapter concerning these functions in the

‘‘ Handbook of Mathematical Functions’’ by Abramowitz and Stegun. [24].

2.1.1 The Airy's Equation

We consider the following homogeneous second order differential equation called the Airy's equation.

$$y'' - xy = 0 \tag{2.1}$$

This differential equation may be solved by the method of Laplace, i.e. in seeking a solution as an integral

$$y = \int_C e^{xz}\, v(z)dz,$$

This is equivalent to solve the first order differential equation

$$v' + z^2 v = 0\,.$$

We thus obtain the solution to the equation (2.1), except a normalization constant

$$y = \int_C e^{xz - z^3/3}\, dz.$$

The integration path C is chosen such that the function $v(z)$ must vanish at the boundaries. This is the reason why the extremities of the path must go into the regions of the complex plane z, where the real part of z^3 is positive (shading regions of the complex plane) [23].

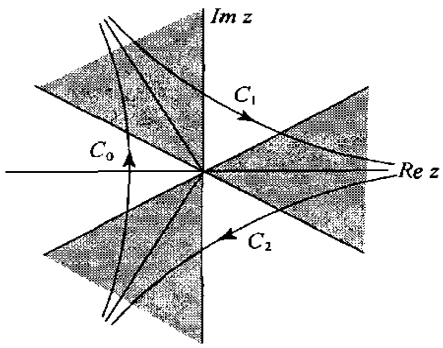


Fig. 2.1 The integration path C [23].

From symmetry considerations, it is useful to work with the paths C_0, C_1 and C_2. Clearly, the integration paths C_1 and C_2 lead to solutions that tend to infinity when x goes to infinity. When we consider the path C_0 and the associated solution, we can deform this curve until it joins the imaginary axis [23]. Now we define the Airy function A_i by

$$A_i = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} e^{xz - z^3/3}\, dz \tag{2.2}$$

If 1, j, j^2 are the cubic roots of unity (that is to say $j = e^{2\pi i/3}$) the functions defined by the paths C_1 and C_2 are respectively the functions $A_i(jx)$ and $A_i(j^2x)$.

We have between these solutions, two by two linearly independent for they satisfy the same second order differential equation, the relation

$$A_i(x) + jA_i(jx) + j^2A_i(j^2x) = 0. \tag{2.3}$$

Now, in place of the functions $A_i(jx)$ and $A_i(j^2x)$, we define the function $B_i(x)$, linearly independent of $A_i(x)$, which has the interesting property to be real when x is real

$$B_i(x) = ij^2A_i(j^2x) - ijA_i(jx)\,. \qquad (2.4)$$

Similarly to $A_i(x)$, we have the relation

$$B_i(x) + jB_i(jx) + j^2B_i(j^2x) = 0. \qquad (2.5)$$

On Figs. 2.2 and 2.3, the plots of the functions $A_i(x)$, $B_i(x)$ and of their derivatives $A_i{}'(x)$ and $B_i{}'(x)$ are given.

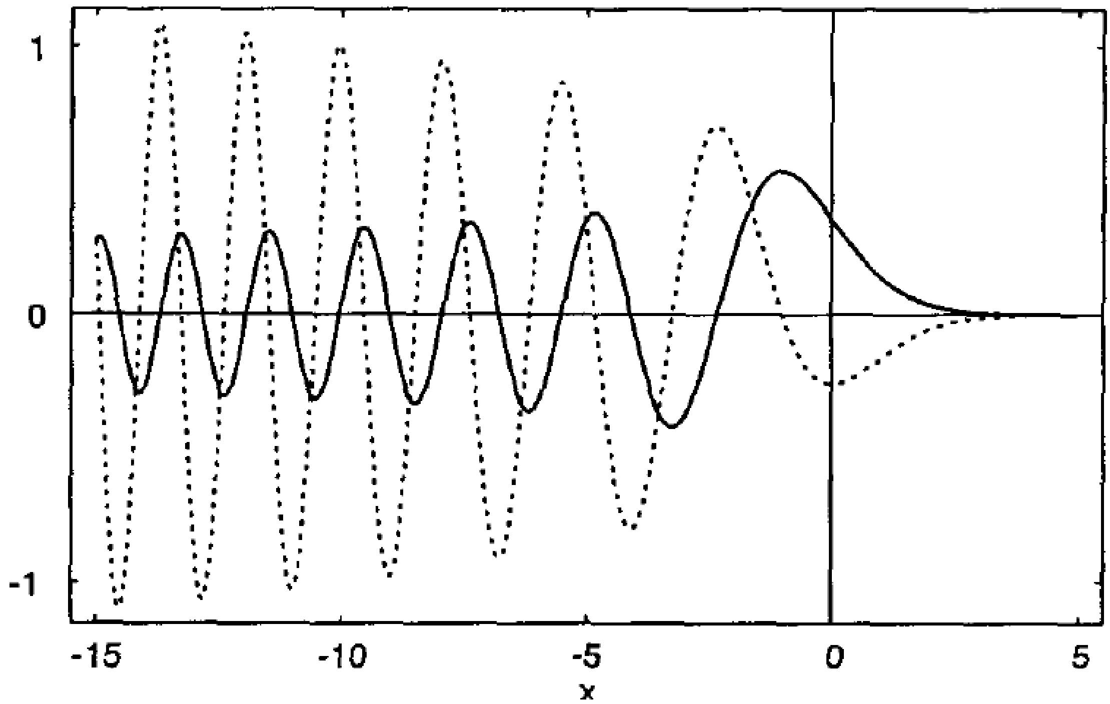


Figure 2.2 Plot of the Airy function ***Ai*** (full line) and its derivative (dotted line) [23].

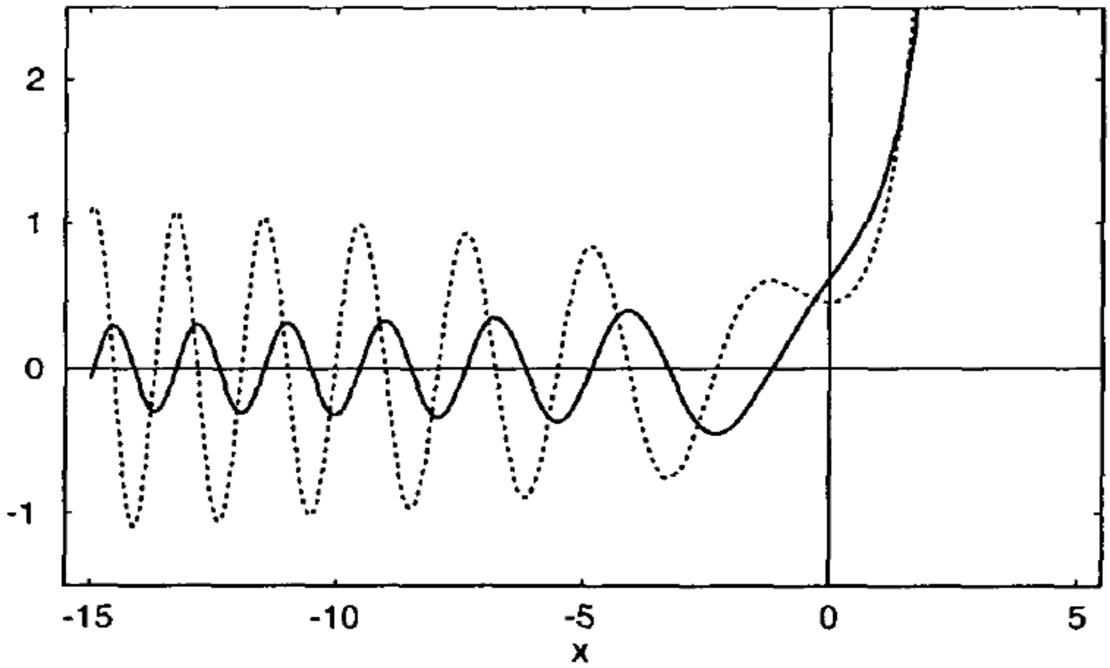


Figure 2.3 Plot of the Airy function ***Bi*** (full line) and its derivative (dotted line)[23].

2.1.2 Wronskians of Homogeneous Airy Functions

The Wronskians,$w\{f,g\}$ of two functions $f(x)$ and $g(x)$ is defined by

$$w\{f,g\} = f(x)\frac{dg(x)}{dx} - \frac{df(x)}{dx}g(x).$$

For the Airy functions $A_i(x)$ and $B_i(x)$, we have the following Wronskians [24].

$$w\{A_i(x), B_i(x)\} = \frac{1}{\pi} \tag{2.6}$$

$$w\{A_i(x), A_i(xe^{2\pi i/3})\} = \frac{e^{-\pi i/6}}{2\pi} \tag{2.7}$$

$$w\{A_i(x),\ A_i(xe^{-2\pi i/3})\} = \frac{e^{\pi i/6}}{2\pi} \tag{2.8}$$

$$w\{A_i(xe^{2\pi i/3}),\ A_i(xe^{-2\pi i/3})\} = \frac{i}{2\pi} \tag{2.9}$$

2.1.3 Particular Values of Airy Functions

The values at the origin of homogeneous Airy functions are

$$A_i(0) = \frac{B_i(0)}{\sqrt{3}} = \frac{1}{3^{2/3}\Gamma(\frac{2}{3})} = 0.355028053887817239 \qquad (2.10)$$

$$-A_i{}'(0) = \frac{B_i{}'(0)}{\sqrt{3}} = \frac{1}{3^{1/3}\Gamma(\frac{1}{3})} = 0.258819403792806798 \qquad (2.11)$$

and therefore

$$A_i(0)A_i{}'(0) = \frac{-1}{2\pi\sqrt{3}} \qquad (2.12)$$

More generally, we havefor the higher derivatives [25]

$$A_i{}^{(n)}(0) = (-1)^n c_n \sin(\pi(n+1)/3) \qquad and \qquad (2.13)$$

$$B_i{}^{(n)}(0) = c_n[1 + \sin(\pi(4n+1)/6) \qquad (2.14)$$

where the coefficient c_n is

$$c_n = \frac{1}{\pi} 3^{(n-2)/3} \Gamma\left(\frac{n+1}{3}\right).$$

2.1.4 Relations Between Airy Functions

The following relations are deduced from the formulae (2.3), (2.4) and (2.5) [26] and [24]

$$A_i(xe^{\pm 2\pi i/3}) = \frac{e^{\pm \pi i/3}}{2}[A_i(x) \pm iB_i(x)] \qquad (2.15)$$

$$A_i{}'\left(xe^{\pm 2\pi i/3}\right) = \frac{e^{\pm \pi i/3}}{2}[A_i{}'(x) \pm iB_i{}'(x)] \qquad (2.16)$$

$$B_i(x) = e^{\pi i/6} A_i\left(xe^{2\pi i/3}\right) + e^{-\pi i/6} A_i\left(xe^{-2\pi i/3}\right) \qquad (2.17)$$

$$B_i{}'(x) = e^{5\pi i/6} A_i{}'\left(xe^{2\pi i/3}\right) + e^{-5\pi i/6} A_i{}'\left(xe^{-2\pi i/3}\right). \qquad (2.18)$$

2.1.5 Integral Representations of AiryFunctions

An integral definition of $A_i(x)$ was given by the formula (2.2). This function can also be defined by the following formulae [24].

$$A_i(x) = \frac{1}{\pi}\int_0^{\infty} \cos\left(\frac{z^3}{3} + xz\right) dz \qquad (2.19)$$

$$A_i(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(z^3/3+xz)}\, dz \qquad (2.20)$$

$$A_i(x) = \frac{x^{1/2}}{2\pi}\int_{-\infty}^{+\infty} e^{ix^{3/2}(z^3/3+z)}\, dz\,, \qquad x > 0 \qquad (2.21)$$

$$A_i(x) = \frac{e^{-\xi}}{2\pi}\int_{-\infty}^{+\infty} e^{-xz^2+iz^3/3}\, dz\,, \qquad x > 0,\ \ \xi = \frac{2}{3}x^{3/2}\,, \qquad (2.22)$$

$$A_i(x) = \frac{e^{-\xi}}{\pi}\int_0^{\infty} e^{-xz^2}\cos(\frac{z^3}{3})\, dz\,, \qquad x > 0,\ \ \xi = \frac{2}{3}x^{3/2}\,. \qquad (2.23)$$

More generally, we have

$$A_i(ax) = \frac{1}{2\pi a}\int_{-\infty}^{+\infty} \exp[i(\frac{u^3}{3a^3} + xu)\, du\,. \qquad (2.24)$$

We give also, the useful formula

$$\int_{-\infty}^{+\infty} \exp[i(\frac{t^3}{3} + at^2 + bt)\, dt = 2\pi e^{ia(2a^2/3-b)} A_i(b - a^2) \qquad (2.25)$$

[27] gives the expressions

$$A_i(x) = \frac{1}{i\pi}\int_0^{i\infty} \cosh(\frac{z^3}{3} - xz)\, dz \qquad (2.26)$$

and for $x > 0$

$$A_i(-x) = \frac{x^{1/2}}{\pi}\int_{-1}^{\infty} \cos[x^{3/2}(\frac{z^3}{3} + z^2 - \frac{2}{3})\, dz, \qquad (2.27)$$

[28] the expression

$$A_i(x) = \frac{e^{-\xi}}{2\pi}\int_0^\infty e^{-x^{1/2}z}\cos(\frac{z^{3/2}}{3})\frac{1}{\sqrt{z}}dz\,, \qquad x > 0\,, \qquad \xi = \frac{2}{3}x^{3/2} \tag{2.28}$$

and [29] the following expression (x>0)

$$A_i(x) = \frac{\sqrt{3}}{2\pi}\int_0^\infty e^{-x\frac{x^3t^3}{3}-\frac{1}{3t^3}}\frac{1}{t^2}dt = \frac{\sqrt{3}}{2\pi}\int_0^\infty e^{-x\frac{t^3}{3}-\frac{x^3}{3t^3}}dt\,. \tag{2.29}$$

For the function B_i, we have the integral representation

$$B_i(x) = \frac{1}{\pi}\int_0^\infty [e^{-z^3/3+xz} + \sin(\frac{z^3}{3} + xz)]\,dz \tag{2.30}$$

Other formulae [30] and [31] having a great interest for the numerical computation of Airy functions are obtained by setting

$$\rho(x) = \frac{1}{\pi^{1/2}2^{11/6}3^{2/3}x^{2/3}}e^{-x}A_i\left[\left(\frac{3x}{2}\right)^{2/3}\right], \qquad x > 0\,. \tag{2.31}$$

In fact, the Bessel function $K_v(x)$ verifies the relation [32]

$$\int_0^\infty \frac{e^{-u}K_v(u)}{u+t}\frac{1}{\sqrt{u}}du = \pi\frac{e^tK_v(t)}{\sqrt{t\cos(\pi v)}}\,, \qquad \mathbb{R}(v) < \frac{1}{2},\ \arg(t) < \pi.$$

In particular, for

$$K_{1/3}(x) = \frac{\pi\sqrt{3}}{(\frac{3\pi}{2})^{1/3}}A_i\left[\left(\frac{3x}{2}\right)^{2/3}\right]$$

we obtain

$$A_i(x) = \frac{e^{-\xi}}{2\pi^{1/2}x^{1/4}}\int_0^\infty \frac{\rho(z)}{1+\frac{z}{\xi}}dz \qquad ,x > 0 \quad ,\xi = \frac{2}{3}x^{3/2} \tag{2.32}$$

with ρ defined as above and $\xi = \frac{2}{3}x^{3/2}$. In a similar fashion, we shall have, for x>0

$$B_i(x) = \frac{e^{\xi}}{\pi^{1/2}x^{1/4}}\int_0^\infty \frac{\rho(z)}{1-\frac{z}{\xi}}dz\,, \tag{2.33}$$

$$A_i(-x) = \frac{1}{\pi^{1/2}x^{1/4}}\int_0^\infty \frac{\cos\left(\xi-\frac{\pi}{4}\right)+\frac{z}{\xi}\sin\left(\xi-\frac{\pi}{4}\right)}{1+\left(\frac{z}{\xi}\right)^2}\rho(z)dz\,, \tag{2.34}$$

$$B_i(-x) = \frac{1}{\pi^{1/2}x^{1/4}}\int_0^\infty \frac{\frac{z}{\xi}\cos\left(\xi-\frac{\pi}{4}\right)-\sin\left(\xi-\frac{\pi}{4}\right)}{1+\left(\frac{z}{\xi}\right)^2}\rho(z)dz\ . \tag{2.35}$$

It should be noted that the integral representations (2.19) and (2.20) are the most frequently used.

2.1.6 Ascending and Asymptotic Series

The expansion of A_i near the origin $x = 0$ is [33] .

$$A_i(x) = \frac{1}{\pi 3^{2/3}}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{n+1}{n!})}{n!}\sin\left[\frac{2}{3}(n+1)\pi\right]\left(3^{1/3}x\right)^n \tag{2.36}$$

The ascending series of $A_i(x)$ and $B_i(x)$ are defined [26] and [24] by the following chain rule

$$A_i(x) = c_1 f(x) - c_2 g(x) \tag{2.37}$$

$$B_i(x) = \sqrt{3}[c_1 f(x) + c_2 g(x)] \tag{2.38}$$

with $c_1 = A_i(0)$ and $c_2 = A_i{}'(0)$, and the series

$$f(x) = \sum_{k=0}^{\infty} 3^k \left(\frac{1}{3}\right)_k \frac{x^{3k}}{(3k)!} = 1 + \frac{1}{3!}x^3 + \frac{1.4}{6!}x^6 + \frac{1.4.7}{9!}x^9 + \cdots$$

$$g(x) = \sum_{k=0}^{\infty} 3^k \left(\frac{2}{3}\right)_k \frac{x^{3k+1}}{(3k+1)!} = x + \frac{2}{4!}x^4 + \frac{2.5}{7!}x^7 + \frac{2.5.8}{10!}x^{10} + \cdots$$

where the Pochammer symbol $(a)_n$ is defined by

$$(a)_0 = 1,\quad (a)_n = \frac{\Gamma\,(a+n)}{\Gamma(a)} = a(a+1)(a+2)\ldots(a+n-1)\ . \tag{2.39}$$

The ascending series of the derivatives $A_i{}'(x)$ and $B_i{}'(x)$ are obtained from the differentiation of the series $f(x)$ and $g(x)$ term-by-term. We obtain therefore

$$A_i{}'(x) = c_1 f'(x) - c_2 g'(x) \tag{2.40}$$

$$B_i{}'(x) = \sqrt{3}[c_1 f'(x) + c_2 g'(x)] \tag{2.41}$$

and the series

$$f'(x) = \frac{1}{2}x^2 + \frac{1}{2.3}\frac{x^5}{5} + \frac{1}{2.3.5.6}\frac{x^8}{8} + \cdots,$$

$$g'(x) = 1 + \frac{1}{1.3}\frac{x^3}{3} + \frac{1}{1.3.4.6}\frac{x^6}{6} + \frac{1}{1.3.4.6.7.9}\frac{x^9}{9} + \cdots \quad .$$

The asymptotic series of A_i and B_i are calculated with the steepest descent method [35] and [36]. We will calculate the asymptotic series of $A_i(x)$ for $x > 0$. The definition (2.2) of A_i allows us to write

$$A_i = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} e^{xt - t^3/3}\, dt \,,$$

where C_0 is the contour defined above . Setting

$$t = \sqrt{x} + iu, \qquad -\infty < u < \infty \;,$$

we obtain

$$\pi e^{\xi} A_i(x) = \int_0^{\infty} e^{-u^2\sqrt{x}} \cos(\frac{u^3}{3})\, du = \frac{1}{2x^{1/4}} \int_{-\infty}^{\infty} e^{-v^2} \cos(\frac{v^3}{3x^{3/4}})\, dv \,,$$

with $\xi = \frac{2}{3}x^{3/2}$. The cosine function may be replaced by its expansion:

$$\pi e^{\xi} A_i(x) = \frac{1}{2x^{1/4}} \int_{-\infty}^{\infty} e^{-v^2} \left(1 - \frac{v^6}{2!\,3^2 x^{3/2}} + \frac{v^{12}}{4!\,3^4 x^3} - \cdots\right) dv \,.$$

Integrating term-by-term

$$\pi e^{\xi} A_i(x) \approx \frac{\pi^{1/2}}{2x^{1/4}} \left(1 - \frac{3.5}{1!\,144 x^{3/2}} + \frac{5.7.9.11}{2!\,144^2 x^3} - \cdots\right),$$

we obtain the formula given below (2.44). The other series are evaluated similarly. For $x \gg 1$ and $s \gg 1$, one defines

$$u_s = \frac{\Gamma(3s + 1/2)}{54^s s!\, \Gamma(s + 1/2)} = \frac{(2s+1)(2s+3)\ldots(6s-1)}{216^s s!} \tag{2.42}$$

$$v_s = -\frac{6s+1}{6s-1}u_s \ , \qquad (2.43)$$

and the other series, according to the notation of [34] .

$$L(z) = \sum_{s=0}^{\infty} \frac{u_s}{z^s} = 1 + \frac{3.5}{1!\,216}\frac{1}{z} + \frac{5.7.9.11}{2!\,216^2}\frac{1}{z^2} + \frac{7.9.11.13.15.17}{3!\,216^3}\frac{1}{z^3} + \cdots$$

$$M(z) = \sum_{s=0}^{\infty} \frac{v_s}{z^s} = 1 - \frac{3.7}{1!\,216}\frac{1}{z} - \frac{5.7.9.13}{2!\,216^2}\frac{1}{z^2} - \frac{7.9.11.13.15.19}{3!\,216^3}\frac{1}{z^3} - \cdots$$

$$P(z) = \sum_{s=0}^{\infty} (-1)^s \frac{u_{2s}}{z^{2s}} = 1 - \frac{5.7.9.11}{2!\,216^2}\frac{1}{z^2} + \frac{9.11.13.15.17.19.21.23}{4!\,216^4}\frac{1}{z^4} - \cdots$$

$$Q(z) = \sum_{s=0}^{\infty} (-1)^s \frac{u_{2s+1}}{z^{2s+1}} = \frac{3.5}{1!\,216}\frac{1}{z} - \frac{7.9.11.13.15.17}{3!\,216^3}\frac{1}{z^3} + \cdots$$

$$R(z) = \sum_{s=0}^{\infty} (-1)^s \frac{v_{2s}}{z^{2s}} = 1 + \frac{5.7.9.13}{2!\,216^2}\frac{1}{z^2} - \frac{9.11.13.15.17.19.21.25}{4!\,216^4}\frac{1}{z^4} + \cdots$$

$$S(z) = \sum_{s=0}^{\infty} (-1)^s \frac{v_{2s+1}}{z^{2s+1}} = -\frac{3.7}{1!\,216}\frac{1}{z} + \frac{7.9.11.13.15.19}{3!\,216^3}\frac{1}{z^3} - \cdots$$

Hence we obtain the asymptotic series of the Airy functions and of their derivatives with $\xi = \frac{2}{3}x^{3/2}$

$$A_i(x) \approx \frac{1}{2\pi^{1/2}x^{1/4}} e^{-\xi} L(-\xi) \ , \qquad (2.44)$$

$$A_i{}'(x) \approx \frac{x^{1/4}}{2\pi^{1/2}} e^{-\xi} M(-\xi) \, , \qquad (2.45)$$

$$B_i(x) \approx \frac{1}{\pi^{1/2}x^{1/4}} e^{\xi} L(\xi) \, , \qquad (2.46)$$

$$B_i{}'(x) \approx \frac{x^{1/4}}{\pi^{1/2}} e^{\xi} M(\xi) \ , \qquad (2.47)$$

$$A_i(-x) \approx \frac{1}{\pi^{1/2}x^{1/4}} \ [\sin\left(\xi - \frac{\pi}{4}\right) Q(\xi) + cos\left(\xi - \frac{\pi}{4}\right) P(\xi) \, , \qquad (2.48)$$

$$A_i'(-x) \approx \frac{x^{1/4}}{\pi^{1/2}} [\sin\left(\xi - \frac{\pi}{4}\right) R(\xi) - cos\left(\xi - \frac{\pi}{4}\right) S(\xi) \ , \tag{2.49}$$

$$B_i(-x) \approx \frac{1}{\pi^{1/2} x^{1/4}} [-\sin\left(\xi - \frac{\pi}{4}\right) P(\xi) + cos\left(\xi - \frac{\pi}{4}\right) Q(\xi) \ , \tag{2.50}$$

$$B_i'(-x) \approx \frac{x^{1/4}}{\pi^{1/2}} [\sin\left(\xi - \frac{\pi}{4}\right) S(\xi) + cos\left(\xi - \frac{\pi}{4}\right) R(\xi) \ . \tag{2.51}$$

2.1.7 Zeros of Airy Functions

Zeros of the Airy function $A_i(x)$ are located on the negative part of the real axis. According to the notation of Miller J.C.P [26], we define a_s and a_s', the s^{th} zeros of $A_i(x)$ and $A_i'(x)$, b_s and b_s' the real zeros of $B_i(x)$ and $B_i'(x)$, β_s and β_s' the complex zeros of $B_i(x)$ and $B_i'(x)$ in the region defined by $\frac{\pi}{3} < \arg(x) < \frac{\pi}{2}$. The complex zeros of $B_i(x)$ and $B_i'(x)$ in the region $-\frac{\pi}{2} < \arg(x) < -\frac{\pi}{3}$ are the conjugates of β_s and β_s'. We thus obtain

$$a_s = -f\left[\frac{3\pi}{8}(4s-1)\right] , \tag{2.52}$$

$$a_s' = -g\left[\frac{3\pi}{8}(4s-3)\right] , \tag{2.53}$$

$$b_s = -f\left[\frac{3\pi}{8}(4s-3)\right] , \tag{2.54}$$

$$b_s' = -g\left[\frac{3\pi}{8}(4s-1)\right] , \tag{2.55}$$

$$\beta_s = e^{i\pi/3} f\left[\frac{3\pi}{8}(4s-1) + \frac{3i}{4}\ln(2)\right] , \tag{2.56}$$

$$\beta_s' = e^{i\pi/3} g\left[\frac{3\pi}{8}(4s-3) + \frac{3i}{4}\ln(2)\right] . \tag{2.57}$$

We also have the relations:

$$A_i'(a_s) = (-1)^{s-1} f_1\left[\frac{3\pi}{8}(4s-1)\right] , \tag{2.58}$$

$$A_i(a_s') = (-1)^{s-1} g_1\left[\frac{3\pi}{8}(4s-3)\right] , \tag{2.59}$$

$$B_i{}'(b_s) = (-1)^{s-1} f_1 \left[\frac{3\pi}{8}(4s-3)\right], \quad (2.60)$$

$$B_i(b_s{}') = (-1)^{s} g_1 \left[\frac{3\pi}{8}(4s-1)\right], \quad (2.61)$$

$$B_i{}'(\beta_s) = (-1)^{s} \sqrt{2} e^{-i\pi/6} f_1 \left[\frac{3\pi}{8}(4s-1) + \frac{3i}{4}\ln(2)\right], \quad (2.62)$$

$$B_i(\beta_s) = (-1)^{s-1} \sqrt{2} e^{i\pi/6} g_1 \left[\frac{3\pi}{8}(4s-3) + \frac{3i}{4}\ln(2)\right]. \quad (2.63)$$

The distribution of the zeros of A_i and B_i in the complex plane is given on Figs. 2.4 and 2.5.

The functions $f(x)$, $g(x)$, $f_1(x)$ and $g_1(x)$ are defined, with $|x| \gg 1$, by the relations

$$f(x) \approx x^{2/3}\left(1 + \frac{5}{48x^2} - \frac{5}{36x^4} + \frac{77125}{82944x^6} - \cdots\right) \quad (2.64)$$

$$g(x) \approx x^{2/3}\left(1 - \frac{7}{48x^2} + \frac{35}{288x^4} - \frac{181223}{207360x^6} + \cdots\right) \quad (2.65)$$

$$f_1(x) \approx \frac{x^{1/6}}{\pi^{1/2}}\left(1 + \frac{5}{48x^2} - \frac{1525}{4608x^4} + \frac{2397875}{663552x^6} - \cdots\right) \quad (2.66)$$

$$g_1(x) \approx \frac{x^{-1/6}}{\pi^{1/2}}\left(1 - \frac{7}{96x^2} + \frac{1673}{6144x^4} - \frac{84394709}{26552080x^6} + \cdots\right) \quad (2.67)$$

zeros of Airy Functions

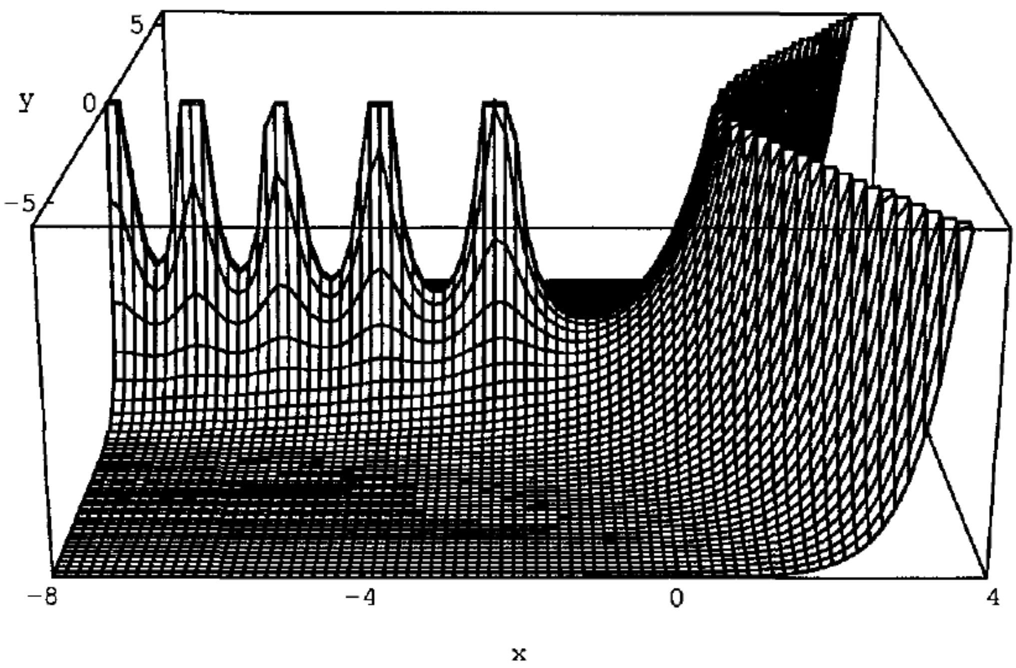


Figure 2.4 Plot of $1/|A_i(x + iy)|$. Zeros of $|A_i(x + iy)|$ are located on the negative part of the real axis. The modulus of the Airy function $|A_i(z)|$ blows up outside this axis, except in the sector defined by $-\frac{\pi}{3} < \arg(z) < \frac{\pi}{3}$ where it goes to 0 [23].

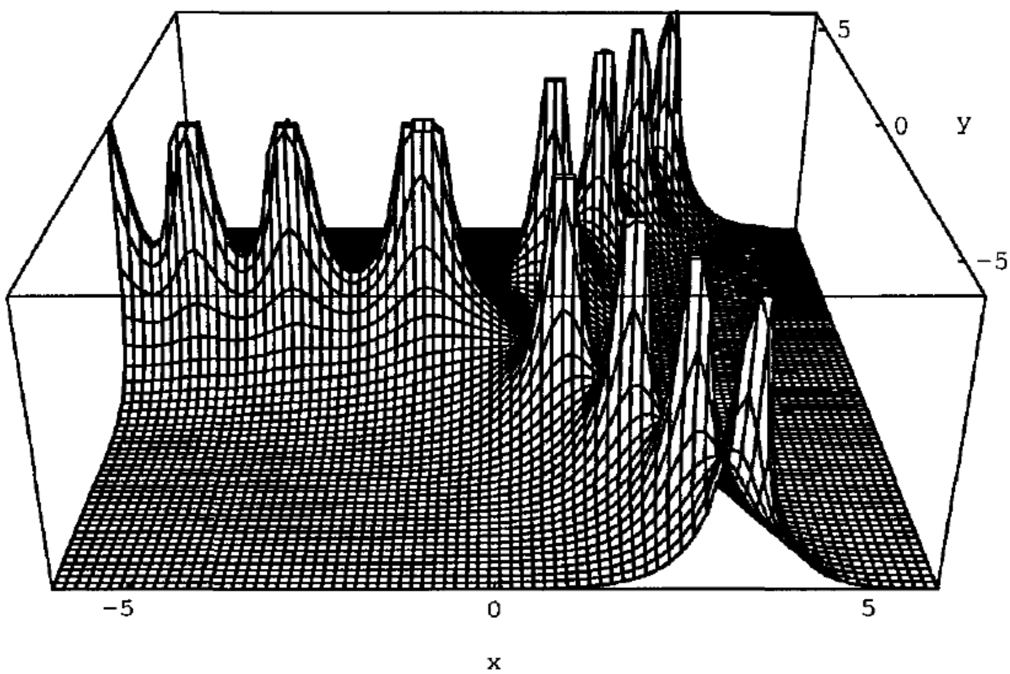


Figure 2.5 Plot of $1/|B_i(x + iy)|$. The real zeros of $|B_i(x + iy)|$ can be discerned on the negative part of the real axis, and the conjugated complex pair of zeros in the sectors $\frac{\pi}{3} < \arg(z) < \frac{\pi}{2}$ and $-\frac{\pi}{2} < \arg(z) < -\frac{\pi}{3}$ [23].

2.1.8 Connection with Bessel Functions

As it is said at the beginnig, Airy functios A_i and B_i may be alternetively written in terms of Bessel functions I and J of $^1/_3$, and of order $^2/_3$ for their derivatives, $\xi = \frac{2}{3} x^{3/2}$ [37], [27].

$$A_i(x) = \frac{\sqrt{x}}{3}\left[I_{-1/3}(\xi) - I_{1/3}(\xi)\right] = \frac{1}{\pi}\sqrt{\frac{x}{3}} K_{1/3}(\xi)\,, \qquad (2.68)$$

$$A_i(-x) = \frac{\sqrt{x}}{3}\left[J_{-1/3}(\xi) - J_{1/3}(\xi)\right] = \sqrt{\frac{x}{3}}\,\Re\left[e^{i\frac{\pi}{6}} H_{1/3}{}^{(1)}(\xi)\right], \qquad (2.69)$$

$$A_i{}'(x) = -\frac{x}{3}\left[I_{-2/3}(\xi) - I_{2/3}(\xi)\right] = -\frac{1}{\pi}\frac{x}{\sqrt{3}} K_2(\xi)\,, \qquad (2.70)$$

$$A_i{}'(-x) == -\frac{x}{3}\left[J_{-2/3}(\xi) - J_{2/3}(\xi)\right] = \frac{x}{\sqrt{3}}\,\Re\left[e^{-i\frac{\pi}{6}} H_{2/3}{}^{(1)}(\xi)\right], \qquad (2.71)$$

$$B_i(x) = \sqrt{\frac{x}{3}}\left[I_{-1/3}(\xi) + I_{1/3}(\xi)\right] = \sqrt{\frac{x}{3}}\,\Re\left[e^{i\frac{\pi}{6}} H_{1/3}{}^{(1)}(-i\xi)\right], \qquad (2.72)$$

$$B_i(-x) = \sqrt{\frac{x}{3}}\left[J_{-1/3}(\xi) - J_{1/3}(\xi)\right] = -\sqrt{\frac{x}{3}}\,\Re\left[e^{i\frac{\pi}{6}} H_{1/3}{}^{(1)}(\xi)\right], \qquad (2.73)$$

$$B_i{}'(x) = \frac{x}{\sqrt{3}}\left[I_{-2/3}(\xi) + I_{2/3}(\xi)\right] = \frac{x}{\sqrt{3}}\,\Re\left[e^{i\frac{\pi}{6}} H_{2/3}{}^{(1)}(-i\xi)\right], \qquad (2.74)$$

$$B_i{}'(-x) = \frac{x}{\sqrt{3}}\left[J_{-2/3}(\xi) + J_{2/3}(\xi)\right] = -\frac{x}{\sqrt{3}}\,\Re\left[e^{-i\frac{\pi}{6}} H_{2/3}{}^{(1)}(\xi)\right]. \qquad (2.75)$$

The modified Bessel function $K_v(z)$ is defined by

$$K_v(z) = \frac{\pi}{2}\frac{I_{-v}(z) + I_v(z)}{\sin(\pi v)}\,,$$

and the Hankel functions $H_v{}^{(1)}(z)$, $H_v{}^{(2)}(z)$ by

$$H_v{}^{(1)}(z) = J_v(z) + iY_v(z)\,,$$

with the Weber function

$$Y_v(z) = \frac{J_v(z) - J_{-v}(z)}{\sin(\pi v)}$$

2.1.9 Power Series Solutions for Airy's Equation

The solutions of the second-order linear differential equation,

$$y'' - ty = 0$$

are called Airy functions. These functions are closely related to the cylinder functions, and play an important role in the theory of asymptotic representations of various special functions arising as solutions of linear differential equations. In particular, the Airy functions turn out to be useful in deriving asymptotic representations of the cylinder functions for large values of |t| and |v|, valid in an extended region of values of t and v, that is the order of the cylinder functions The Airy functions also have a variety of applications to mathematical physics, e.g., the theory of diffraction of radio waves around the earth's surface [38].

We now present the power series solutions of Airy functions.

Airy's Equation

$$y'' - ty = 0$$

which is used in physics to model the defraction of light.

We want to find power series solutions for this second-order linear differential equation. The generic form of a power series is

$$y(t) = \sum_{n=0}^{\infty} a_n\, t^n$$

We have to determine the right choice for the coefficient (a_n). Recall that

$$y''(\mathrm{t}) = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}$$

Plugging this information into the differential equation we obtain:

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - t\sum_{n=2}^{\infty} a_n t^n = 0$$

or equivalently

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - \sum_{n=2}^{\infty} a_n t^{n+1} = 0$$

Our next goal is to simplify this expression such that only one summation sign remains. The obstacle we encounteris that the powers of both sums are different, t^{n-2} for the first sum and t^{n+1} for the second sum. We make them the same by shifting the index of the first sum up by 2 unitsand the index of the first sum up by 2 units and the index of the second sum down by one unit to obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n - \sum_{n=1}^{\infty} a_{n-1} t^n = 0$$

Now run into the next problem; the second sum starts at n=1, while the first sum has one more term and starts at n=0. We split off the 0th term of the first sum:

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n = 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2} t^n$$

Now we can combine the two sums as follows:

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2} t^n - \sum_{n=1}^{\infty} a_{n-1} t^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - a_{n-1}] t^n = 0$$

The power series on the left is identically equal to zero, consequently all of its coefficients are equal to zero.

For all $n = 1,2,3, \ldots$ we can slightly rewrite as

$$2a_2 = 0$$

$$(n+2)(n+1)a_{n+2} - a_{n-1} = 0$$

For all $n = 1,2,3, \ldots \ldots$

$$2a_2 = 0$$

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}$$

These equations are known as the ‘‘recurrence relations’’ of the differential equations.

The recurrence relations permit us to compute all coefficients in terms of a_0 and a_1.

We already know from the 0th recurrence relation that $a_2 = 0$. Let’s compute a_3 by reading off the recurrence relation for n=1.

$$a_3 = \frac{a_0}{2.3}$$

$$a_4 = \frac{a_1}{3.4}$$

$$a_5 = \frac{a_2}{4.5} = 0$$

$$a_6 = \frac{a_3}{5.6} = \frac{a_0}{(2.3).(5.6)}$$

$$a_7 = \frac{a_4}{6.7} = \frac{a_1}{(3.4).(6.7)}$$

$$a_8 = \frac{a_5}{7.8} = 0$$

$$a_9 = \frac{a_6}{8.9} = \frac{a_0}{(2.3).(5.6)(8.9)}$$

The hardest part, as usual, is to recognize the patterns evolving: in this case we have to consider three cases:

All the terms a_2 , a_5, a_8, … are equal to zero. We can write this in compact form as

$$a_{3k+2} = 0 \ \ for\ all\ k = 0,1,2,3, ...$$

All the terms a_3 , a_6, a_9, … are multiples of a_0 .We can be more precise:

$$a_{3k} = \frac{1}{(2.3).(5.6) ..... ((3k-1).(3k))} . a_0 \ \ for\ all\ k = 1,2,3, ...$$

All the terms a_4 , a_7, a_{10}, … are multiples of a_1. We can be more precise:

$$a_{3k+1} = \frac{1}{(3.4).(6.7)\ldots..((3k).(3k+1))}.a_1 \quad for\ all\ k = 1,2,3,\ldots$$

Thus the general form of the solutions to Airy's Equation is given by

$$y(t) = a_0\left(1 + \sum_{k=1}^{\infty}\frac{t^{3k}}{(2.3).(5.6)\ldots..\big((3k-1).(3k)\big)}\right) + a_1\left(t + \sum_{k=1}^{\infty}\frac{t^{3k+1}}{(3.4).(6.7)\ldots..\big((3k).(3k+1)\big)}\right)$$

Note that, as always, $y(0) = a_0$ and $y'(0) = a_1$. Thus it is trivial to determine a_0 and a_1 when you want to solve an initial value problem.

In particular,

$$y_1(t) = 1 + \sum_{k=1}^{\infty}\frac{t^{3k}}{(2.3).(5.6)\ldots..\big((3k-1).(3k)\big)} \quad and$$

$$y_2(t) = t + \sum_{k=1}^{\infty}\frac{t^{3k+1}}{(3.4).(6.7)\ldots..\big((3k).(3k+1)\big)}$$

forma fundamental system of solutions for Airy's Differential Equations. Below you see a picture of these two solutions. Note that for negative t, the solutions behave somewhat like the oscillating solutions of

$$y'' + y = 0\ ,$$

while for positive t, they behave somewhat like the exponential solutions of the differential equation

$$y'' - y = 0\ .$$

Power Series Solutions for Airy's Equation

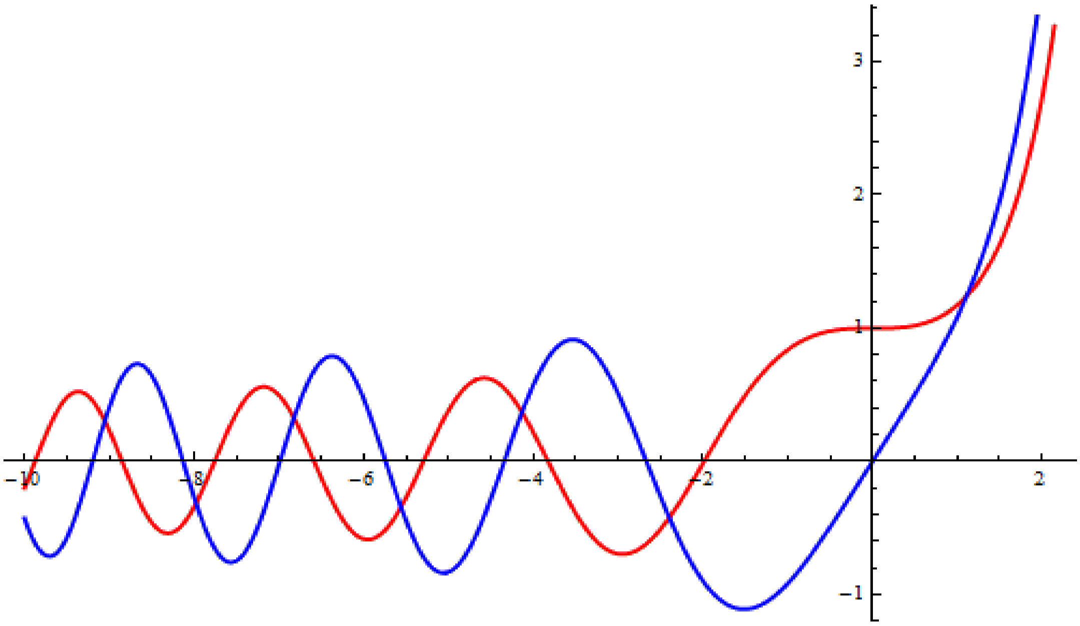


Figure 2.6 Plot of the series solutions, $y_1(t)$ and $y_2(t)$ of Airy's Equation [38].

2.1.10 The Radius of Convergence of the Power Series Solutions

Airy's Equation:

$y'' - ty = 0.$

$$y_1(t) = 1 + \sum_{k=1}^{\infty} \frac{t^{3k}}{(2.3).(5.6)\ldots\ldots((3k-1).(3k))} \quad and$$

$$y_2(t) = t + \sum_{k=1}^{\infty} \frac{t^{3k+1}}{(3.4).(6.7)\ldots\ldots((3k).(3k+1))}$$

form a fundamental system of solutions for Airy's Differential Equation. The natural questions arise, for which values of t these series converge, and for which of t these series solve the differential equation.

The first question could be answered by finding the radius of convergence of the power series, but it turns out that there is an elegant theorem, due to Lazarus Fuchs (1833-1902) which solves both of these questions simultaneously [39] and [40].

Fuchs's Theorem Consider the differential equation

$$y'' + p(t)y' + q(t)y = 0$$

with initial conditions of the form

$$y(0) = y_0 \; and \; y'(0) = y'_0.$$

Let r>0. If both p(t) and q(t) have Taylor series, which converge on the interval (-r, r) then the differential equation has a unique power series solution y(t), which also converges on the interval (-r, r).

In other words, the Radius of convergence of the series solution is at least as big as the minimum of the radii of convergence of p(t) and q(t).

In particular, if both p(t) and q(t) are polynomials then y(t) solves the differential equation for all $t \in \mathbb{R}$.

Let us look at some other examples.

Hermite's Equation of order n has the form

$$y'' - 2ty' + 2ny = 0$$

Where n is usually a non – negative integer as in the case of Airy's Equation, both $p(t) = -2t$ and $q(t) = 2n$ are polynomials, thus Hermite's Equation has power series solutions which converse and solve the differential equation for all $t \in \mathbb{R}$.

Legendre's Equation of order α has the form

$$(1 - t^2) - 2ty' + \alpha \,.\, (\alpha + 1)y = 0$$

where α is a real number.

We have to rewrite this equation to be able to apply Fuchs's Theorem. Let's divide by $(1 - t^2)$;

$$y'' - \frac{2t}{1 - t^2} y' + \frac{\alpha(\alpha + 1)}{1 - t^2} y = 0$$

Now the coefficient in front of y'' is 1 as required.

What is the radius of convergence of the power series representations of

$$p(t) = -\frac{2t}{1 - t^2} \quad and \quad q(t) = \frac{\alpha(\alpha + 1)}{1 - t^2}$$

(The center as in all our examples will be t=0.) We really have to investigate this question only for the function

$$f(t) = \frac{1}{1-t^2}$$

Since multiplication by a polynomial $(-2t, and\ \alpha(\alpha+1), respectively)$ does not change the radius of convergence. The geometric series :

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Converges when -1<x<1 lf we substitute $x = t^2$, we obtain the power series representation we seek :

$$f(t) = \frac{1}{1-t^2} = \sum_{n=0}^{\infty} t^{2n}$$

which will be convergent when $-1 < x = t^2 < 1$ $, i.e$ when $-1 < t < 1$. Thus both

$$p(t) = -\frac{2t}{1-t^2} \quad and$$

$$q(t) = \frac{\alpha(\alpha+1)}{1-t^2}$$

will converge on the interval (-1,1).

Consequently, by Fuchs's result, series solutions to Legendre's Equation will converge and solve the equation on the interval (-1,1).

Bessel's Equation of order α has the form

$$t^2 y'' + t y' + (t^2 - \alpha^2) y = 0$$

where α is a non-negative real number.

Once again we have to be careful! Let's divide by t^2:

$$y'' + \frac{1}{t} y' + \frac{t^2 - \alpha^2}{t^2} y = 0$$

Now the coefficient in front of y'' is 1 as required by Fuchs's Theorem.

The function $p(t) = \frac{1}{t}$ has a singularity at t=0, thus p(t) fails to have a Taylor series with center t=0.

Consequently, Fuchs's result does not even guarantee the existence of power series solutions to Bessel's equation. As it turns out, Bessel's Equation does indeed not always have solutions, which can be written as power series. Nevertheless, there is a method similar to the one presented here to find the solutions to Bessel's Equation.

2.2 General Properties of the Time-Space Modes

Here, some properties of the time-space modes have been given to clarify the characteristics expressed in some steps.

2.2.1 Completeness of the Time-Space Modes

Both TE (1.7) and TM (1.11) time-domain modes define the arbitrary pair of X_1, X_2 which are determined by the following inner product in real valued functional space that has 6-component.

$$(X_1, X_2) = \frac{1}{S}\int_S (\varepsilon_0 \mathbf{E}_1 \mathbf{E_2} + \mu_0 \mathbf{H}_1 \mathbf{H_2}) ds < \infty \qquad (2.76)$$

where, $X_1 = \mathrm{col}(\boldsymbol{E}_1, \boldsymbol{H}_1)$ and $X_2 = \mathrm{col}(\boldsymbol{E}_2, \boldsymbol{H}_2)$ and col. stands for ”column” and ”.”is the scalar product for 3-component vectors. The set of TE (1.7) and TM (1.11) time-domain modes have been shown as following;

$$\boldsymbol{G} = \{X_m^h\}_{m=0}^{\infty} \boldsymbol{J} = \{X_m^e\}_{m=1}^{\infty} \qquad (2.77)$$

where $X_m^h = col(\boldsymbol{E}_m^h, \boldsymbol{H}_m^h)$ and $X_m^e = col(\boldsymbol{E}_m^e, \boldsymbol{H}_m^e)$.

For $m = m^{'}$, if the first pair of distinct elements X_m^h and $X_{m'}^h$ are replaced insted of X_1 and X_2 in (2.76). $(X_m^h, X_{m'}^h) = 0$ is obtained.

This shows us that all elements in the set of $\boldsymbol{G}$ are mutually orthogonal. The same is also true for the elements of the setof $\boldsymbol{J}$.

Now, take an X_m^h element from the set $\boldsymbol{G}$ and an $X_{m'}^e$ element from the set $\boldsymbol{J}$ then put them into equation (2.76) to their places. m and $m^{'}$ to be arbitrary, $(X_m^h, X_{m'}^e) = 0$ is obtained. The completeness of each $\boldsymbol{G}$ and $\boldsymbol{J}$ in itself has been proven in the studies of Tretyakov [16], [6].

In addition, $\boldsymbol{G}$ and $\boldsymbol{J}$ form a subspace in the solutions space. Finally, their direct sum is $\boldsymbol{G} \oplus \boldsymbol{J}$. Therefore, one of the subspace is orthogonal to the other.

2.2.2 Relativistic Invariance of the Time-Space Modes

Every time-space $(\boldsymbol{E}_m, \boldsymbol{H}_m)$ modal field of the equations (1.7) and (1.11) is a certain solution of the system of Maxwell's equations with time derivative ∂_{t}:

$$\nabla \times \mathbf{E}_m = -\mu_0\, \partial_{\mathrm{t}} \mathbf{H}_m \tag{2.78}$$

$$\nabla \times \mathbf{H}_m = \varepsilon_0 \partial_t \mathbf{E}_m$$

The time-space modal fields which are the solution of Maxwell's equations with time derivative ∂_{t} should be calculated according to the sepecial theory of relativity.

Let us take the inertia frame of the time-space modes of equations (1.7) and (1.11) as (r, z, t) and F as the function of t. Then get a new reference frame F' and assume the motion with a constant velocity v, along the oz axis. Let's show the time and coordinates of F' as (r', z', t'). The relationship between (r, z, t) and (r', z', t') is determined directly by Lorentz transformation as follows;

$$r = r', \qquad z = (z' + vt')\gamma\,, \qquad t = \left(t' + \frac{vz'}{c^2}\right)\gamma \tag{2.79}$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\beta = \frac{v}{c}$.

The inverse Lorentz transformation is obtained symmetrically as follows;

$$r = r', \qquad z' = (z - vt)\gamma\,, \qquad t' = \left(t - \frac{vz}{c^2}\right)\gamma \tag{2.80}$$

when $r = r'$ has been taken, on the contrary, the solutions in the reference frames F and F' of the problems (1.5) and (1.10) overlap each other. This means, physically, the waveguide modal bases in the reference frames F and F' are the same. That is, the waveguide modal fields in the reference frames F and F' are invariant. Evidently, the eigenvalues v_m^2 and κ_m^2 are invariant in these inertia reference frames [9].

Lastly, the TE and TM modes in the time-space are uniform in these reference frames [9].

Let us choose τ and ξ as the dimensionless time and axial coordinate as in (1.14), respectively.

For the TE modes; $\tau = \nu_m ct$ and $\xi = \nu_m z$.

For the TM modes; $\tau = \kappa_m ct$ and $\xi = \kappa_m z$.

Then, the equations (1.9) and (1.12) are the same. The introduction of invariance of amplitudes $h_m(\xi,\tau)$ and $e_m(\xi,\tau)$ which are obtained from Klein-Gordon equations are with the help of Klein-Gordon equation (1.15).

$$\left(\partial_\tau^2 - \partial_\xi^2 + 1\right) f(\xi,\tau) = 0$$

İf $\tau = \nu_m ct$ and $\xi = \nu_m z$, then $f(\xi,\tau)$ is instead of $h_m(\xi,\tau)$

İf $\tau = \kappa_m ct$ and $\xi = \kappa_m z$, then $f(\xi,\tau)$ is instead of $e_m(\xi,\tau)$.

It can be shown that the equation (1.15) has the same form in both reference frames F and F'. To illustrate this, take the Lorentz transformations in (2.79), (2.80) and adapt them to variables (ξ,τ), (ξ',τ'):

$$\xi = (\xi' + \beta\tau')\gamma \ , \ \ \tau = (\tau' + \beta\xi')\gamma$$

$$\xi' = (\xi - \beta\tau)\gamma \ , \ \ \tau' = (\tau - \beta\xi)\gamma \tag{2.81}$$

Take the solution of the equation (1.15) in new variables (ξ',τ') instead of the old variables (ξ,τ). With the help of the inverse Lorentz transformation, determine the $f[\xi'(\xi,\tau),\tau'(\xi,\tau)]$ instead of $f(\xi,\tau)$ and reinterpret this: Take the partial derivatives $\partial_{\tau'}$ and $\partial_{\xi'}$.

$$\partial_\tau f(\xi',\tau') = \left(-\beta\gamma\partial_{\xi'} + \gamma\partial_{\tau'}\right) f(\xi',\tau'),$$

$$\partial_\xi f(\xi',\tau') = \left(\gamma\partial_{\xi'} - \beta\gamma\partial_{\tau'}\right) f(\xi',\tau'). \tag{2.82}$$

By repeating this twice in equation (1.15);

$$\left(\partial_{\tau'}^2 - \partial_{\xi'}^2 + 1\right) f(\xi',\tau') = 0 \tag{2.83}$$

is obtained. The equation (2.83) is valid in the reference frame F'. It has the same form with the equation (2.81) in the reference frame F [9].

2.2.3 Energy Conservation for the Time-Space Modes

Let us apply Poynting theorem to the equation (2.78) and find an average result on the surface S, the cross- sectional area of the waveguide. Average Poynting vector and average modal field energy density are shown with P_z and W, respectively. Taking the notations in terms of the equation (1.15), the result is given here:

$$W(\xi,\tau) = \frac{[(-\partial_\tau f)^2 + (\partial_\xi f)^2 + f^2]}{2} \tag{2.84}$$

$$P_z(\xi,\tau) = c(-\partial_\tau f)(\partial_\xi f)$$

So, Poynting theorem

$$\frac{\partial_\xi[-(\partial_\tau f)(\partial_\xi f)] - \partial_\tau[(-\partial_\tau f)^2 + (\partial_\xi f)^2 + f^2]}{2} = 0 \tag{2.85}$$

is obtained.

2.2.4 Initial Conditions for Klein-Gordon Equations

Klein-Gordon equation should be supported with a pair of initial conditions like any other second order partial differential equations (PDE). Physically, they play a role in excitation of an appropriate signal source.

It is assumed that the source isn't excitated before the time $t = 0$, it means there is no motion, but at $t = 0$ the exitation is started. If so, the initial conditions should be written as,

$$f(\xi,\tau)|_{\xi=0} = \begin{cases} \varphi(\tau), & \tau \geq 0 \quad \Rightarrow for\ t \geq 0 \\ 0, & \tau < 0 \quad \Rightarrow for\ t < 0 \end{cases}$$

(2.86)

$$\left.\frac{\partial}{\partial\tau} f(\xi,\tau)\right|_{\xi=0} = \begin{cases} \hat{\varphi}(\tau), & \tau \geq 0 \quad \Rightarrow for\ t \geq 0 \\ 0, & \tau < 0 \quad \Rightarrow for\ t < 0 \end{cases} .$$

$\varphi(\tau)$, $\hat{\varphi}(\tau)$ should be given and

$\xi = 0 \ \Rightarrow z = 0$ should be.

2.2.5 The Principle of Causality

The solutions of Klein-Gordon equation adhere to the requirements of the causality principle. Here are two comments and they support one another. If the sources

are zero at the starting time, there is a weak causality, that means that all the fields are zero. In our problem, this corresponds to the case $\tau < 0$. The strong causality condition follows the Einstein's postulate which suggests that any magnetic field radiates signal with the speed of light, c in the space.

In our problem, the source is located in the cross-sectional area of the waveguide at $\xi = 0$. This means that the solution of Klein-Gordon equation must be zero beyond the source point $\xi = 0$, after $\xi = \tau$, i.e; $z = ct$.

Therefore, if the signal spreads along the oz axis, then the solution of Klein-Gordon equation should be read physically as follows:

$$f(\xi,\tau) = \begin{cases} f(\xi,\tau) = 0\,, & if\ \tau < 0 \\ f(\xi,\tau) \neq 0, & if\ 0 \leq \xi \leq \tau \\ f(\xi,\tau) = 0, & if\ \xi > \tau. \end{cases} \tag{2.87}$$

CHAPTER 3

STATEMENT OF THE PROBLEM

Before starting to study the problem, some definitions and expressions used in the study have been defined and the problem has been stated in this section.

3.1 Definition of the Waveguide

An empty ideal waveguide which has a constant cross-sectional area S that doesn't change along oz axis will be considered (j=0).

The contour L is assumed to be arbitrary, but sufficiently smooth. Sufficiently smooth means that none of the inner angles measured in S exceeds π. In the rest of the study the mutually orthogonal unit vectors (**z**,**l**,**n**) ,e.i. $\boldsymbol{z} \times \boldsymbol{l} = \boldsymbol{n}$, $\boldsymbol{l} \times \boldsymbol{n} = \boldsymbol{z}$, $\boldsymbol{n} \times \boldsymbol{z} = \boldsymbol{l}$ have been used. The vector **n** is the outer normal to the surface S, the vector**z**is oriented along the oz axis and the vector **l** is tangent to the contuor L.

3.2 Standard Formulation of Maxwell's Equations

For **E** electric and **H** magnetic field vectors the following vectorial Maxwell's equations will be solved.

$$\nabla \times \mathbf{E}(\mathbf{R},\mathrm{t}) = -\mu_0\, \partial_\mathrm{t}\mathbf{H}(\mathbf{R},\mathrm{t})$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{R},t) = \varepsilon_0 \partial_t \boldsymbol{E}(\boldsymbol{R},t) \qquad (3.1)$$

In addition to (3.1), the following scalar Maxwell equations will also be used.

$$\nabla.\, \mathbf{E}(\mathbf{R},\mathrm{t}) = 0$$

$$\nabla.\, \boldsymbol{H}(\boldsymbol{R},t) = 0 \qquad (3.2)$$

The equations (3.1) and (3.2) are valid in the waveguide, except the surface. Assuming that the surface area of the waveguide has property of excellent electical conductivity property and the components of the fields are exposing to the following boundary conditions.

$$\boldsymbol{n}.\boldsymbol{H}(\boldsymbol{R},t)|_L = 0, \quad \boldsymbol{l}.\boldsymbol{E}(\boldsymbol{R},t)|_L = 0, \quad \boldsymbol{z}.\boldsymbol{E}(\boldsymbol{R},t)|_L = 0 \tag{3.3}$$

The Maxwell's equations are hyperbolic partial differential equations, so the equations (3.1) should be subjected to the given initials conditions.

$$\mathbf{E}(\mathbf{R},0) = 0, \;\; \boldsymbol{H}(\boldsymbol{R},0) = 0 \tag{3.4}$$

The electromagnetic field energy is finite due to physical principles. So the initial-boundary value problems (3.1)-(3.4) must be solved in class of integrable vector functions. Therefore, the energy characteristics for **E** and ***H*** should be introduced. First, it is useful to give the poynting theorem.

For electromagnetic waves, there is a mathematical description of the energy transfer rate in per unit time. The differential form of energy conservation will be obtained with this theorem. In vector analysis,

$$\nabla.(\mathbf{E}\times\mathbf{H}) = \mathbf{H}.(\nabla\times\mathbf{E}) - \mathbf{E}.(\nabla\times\mathbf{H})$$

is known . When the Maxwell's equations in (3.1) are substituted in this given equality,

$$\nabla.(\mathbf{E}\times\mathbf{H}) = -\mu_0\mathbf{H}\partial_{\boldsymbol{t}}\mathbf{H} \;\; -\varepsilon_0\mathbf{E}\partial_{\boldsymbol{t}}\mathbf{E}$$

is obtained. From the product rule for the derivative

$$\mathbf{H}\partial_{\boldsymbol{t}}\mathbf{H} = \frac{1}{2}\frac{\partial(\mathbf{H}.\mathbf{H})}{\partial t} = \frac{1}{2}\frac{\partial|\mathbf{H}|^{\mathbf{2}}}{\partial t}$$

$$\mathbf{E}\partial_{\boldsymbol{t}}\mathbf{E} = \frac{1}{2}\frac{\partial(\mathbf{E}.\mathbf{E})}{\partial t} = \frac{1}{2}\frac{\partial|\mathbf{E}|^{\mathbf{2}}}{\partial t}$$

After substitution of these,

$$\nabla.(\mathbf{E}\times\mathbf{H}) = -\frac{\partial}{\partial t}(\frac{1}{2}\mu_0|\mathbf{H}|^{\mathbf{2}} + \frac{1}{2}\varepsilon_0|\mathbf{E}|^{\mathbf{2}})$$

can be obtained as the point form of Poynting's theorem.

Lastly, let's integrate over the volume V of the closed surface S.

$$\int_v \nabla.(\mathbf{E}\times\mathbf{H})\,dv = \int_v -\frac{\partial}{\partial t}(\frac{1}{2}\mu_0|\mathbf{H}|^2 + \frac{1}{2}\varepsilon_0|\mathbf{E}|^2)\,dv$$

Before applying the divergence theorem, let’s state the theorem:

Divergence Theorem Let S be a closed surface bounding a region of volume V. Choose the outward drawn normal to the surface as the positive normal. If **F** is continuous and has continuous partial derivatives in the region then

$$\int_v \nabla.(\mathbf{F})\,dv = \oint_s (\boldsymbol{F}.\boldsymbol{n})ds$$

In words this theorem, called the divergence theorem or Green’s theorem in space, states that the surface integral of the normal component of a vector **F** taken over a closed surface is equal to the integral of the divergence of **F** taken over the volume enclosed by the surface.[46]

if the divergence theorem is applied to the left hand side of the equality above,

$$\oint_s (\mathbf{E}\times\mathbf{H})\,ds = -\frac{\partial}{\partial t}\int_v (\frac{1}{2}\mu_0|\mathbf{H}|^2 + \frac{1}{2}\varepsilon_0|\mathbf{E}|^2)\,dv$$

can be obtained as the integral form of the Poynting’s theorem. The right hand side of this equality is interpreted as a decrease in the magnetic and electric power stored in the volume V. Due to energy conservation, the statement given by the right hand side of this equality must equalize the power drops out the volume along the limited surface S. As a result the Poynting vector

$$\boldsymbol{P} = \boldsymbol{E}\times\boldsymbol{H}$$

written in the left hand side, must show the transferred power of electromagnetic field leaving the volume V per unit area. Energy characteristics for vector fields $\boldsymbol{E}$ and $\boldsymbol{H}$ are defined as ,

$$W(z,t) = \frac{1}{2S}\int_S \{\varepsilon_0(\boldsymbol{E}.\boldsymbol{E}) + \mu_0(\boldsymbol{H}.\boldsymbol{H})\}ds$$

$$P_z(z,t) = \frac{1}{S}\int_S \boldsymbol{z}.[\boldsymbol{E}\times\boldsymbol{H}]ds \qquad (3.5)$$

where $P_z(z,t)$ is the z-component of the Poynting vector and $W(z,t)$ is the energy stored in the field. When the Poynting's theorem is applied to Maxwell's equations in (3.1), in time domain the differential form of the energy conservation will be obtained as

$$\partial_z P_z(z,t) + \partial_t W(z,t) = 0 \ .$$

3.3 Transverse – Longitudinal Decompositions

The position vector $\boldsymbol{R}$ and ∇ operator can be decomposed as;

$$\boldsymbol{R} = \boldsymbol{r} + \mathbf{z}z, \quad \nabla = \nabla_\perp + \mathbf{z}\partial_z.$$

where $\boldsymbol{r}$ is the projection of the vector $\boldsymbol{R}$ onto the domain S indicating a point of observation within S and the differential procedure $\nabla_\perp$ acts on the transverse coordinates $(\boldsymbol{r})$.

The field vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ can be decomposed as;

$$\boldsymbol{E}(\boldsymbol{R},t) = \boldsymbol{E}(\boldsymbol{r},z,t) + \mathbf{z}E_z(\boldsymbol{r},z,t)$$

$$\boldsymbol{H}(\boldsymbol{R},t) = \boldsymbol{H}(\boldsymbol{r},z,t) + \mathbf{z}H_z(\boldsymbol{r},z,t)$$

As their transverse and longitudinal coordinates where $\boldsymbol{E}(\boldsymbol{r},z,t)$ and $\boldsymbol{H}(\boldsymbol{r},z,t)$ are the projections of the electric and magnetic field vectors onto the domain S, respectively, E_z and H_z are the magnitudes of z-components of the same vector fields.

The Maxwell's equations in (3.1) can be decomposed as their transverse and longitudinal parts using the above data. Then the equation (3.2) will be added to them.

$$\nabla\times\mathbf{E}(\mathbf{R},\mathrm{t}) = [(\nabla_\perp + \mathbf{z}\partial_z)\times(\boldsymbol{E} + \mathbf{z}E_z)]$$

$$\nabla\times\boldsymbol{H}(\boldsymbol{R},t) = [(\nabla_\perp + \mathbf{z}\partial_z)\times(\boldsymbol{H} + \mathbf{z}H_z)]$$

To find the left-hand sides of the above equations, it is introduced a three-component vector field $\boldsymbol{\mathcal{A}}$, and calculate the curl of the vector as shown below.

$$\nabla\times\boldsymbol{\mathcal{A}} = [(\nabla_\perp + \mathbf{z}\partial_z)\times(\boldsymbol{A} + \mathbf{z}A_z)]$$

$$= [(\nabla_{\perp} \times \mathbf{A}) + (\nabla_{\perp} \times \mathbf{z}A_z)] + [\mathbf{z}\partial_z \times \mathbf{A}] + [\mathbf{z}\partial_z \times \mathbf{z}A_z]$$

It is evident that

$[\mathbf{z}\partial_z \times \mathbf{z}A_z] = 0, (\nabla_{\perp} \times \mathbf{z}A_z) = (\nabla_{\perp}A_z \times \mathbf{z}) = (\nabla_{\perp} \times \mathbf{z})A_z$ and

$[\mathbf{z}\partial_z \times \mathbf{A}] = \partial_z[\mathbf{z} \times \mathbf{A}]$, thus

$$\nabla \times \boldsymbol{\mathcal{A}} = [(\nabla_{\perp} + \mathbf{z}\partial_z) \times (\boldsymbol{A} + \mathbf{z}A_z)] = [(\nabla_{\perp} \times \mathbf{A}) + (\nabla_{\perp} \times \mathbf{z})A_z] + \partial_z[\mathbf{z} \times \mathbf{A}]$$

is obtained. We now can apply this result to the transverse and longitudinal parts of the Maxwell's equations as given below;

$$\nabla \times \mathbf{E}(\mathbf{R}, \mathrm{t}) = [(\nabla_{\perp} + \mathbf{z}\partial_z) \times (\boldsymbol{E} + \mathbf{z}E_z)]$$

$$= [(\nabla_{\perp} \times \mathbf{E}) + (\nabla_{\perp} \times \mathbf{z})E_z] + \partial_z[\mathbf{z} \times \mathbf{E}] = -\mu_0\partial_t\mathbf{H} - \mathbf{z}\mu_0\partial_t H_z$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{R}, t) = [(\nabla_{\perp} + \mathbf{z}\partial_z) \times (\boldsymbol{H} + \mathbf{z}H_z)]$$

$$= [(\nabla_{\perp} \times \mathbf{H}) + (\nabla_{\perp} \times \mathbf{z})H_z] + \partial_z[\mathbf{z} \times \mathbf{H}] = \varepsilon_0\partial_t\mathbf{E} + \mathbf{z}\varepsilon_0\partial_t E_z$$

By making use of a well-known vector identity $\boldsymbol{A}.(\boldsymbol{B}X\boldsymbol{C}) = \boldsymbol{B}.(\boldsymbol{C}X\boldsymbol{A})$, the scalar multiplications of above equations by the unit vector **z** yield ;

$$\boldsymbol{z}.[\nabla \times \mathbf{E}(\mathbf{R}, \mathrm{t})] = \boldsymbol{z}.[\nabla_{\perp} \times \mathbf{E}] = -\nabla_{\perp}.[\boldsymbol{z} \times \mathbf{E}] = -\mu_0\partial_t H_z,$$

$$\boldsymbol{z}.[\nabla \times \mathbf{H}(\mathbf{R}, \mathrm{t})] = \boldsymbol{z}.[\nabla_{\perp} \times \mathbf{H}] = \nabla_{\perp}.[\boldsymbol{H} \times \boldsymbol{z}] = \varepsilon_0\partial_t E_z.$$

The projection of $\nabla \times \mathbf{E}(\mathbf{R}, \mathrm{t})$ and $\nabla \times \boldsymbol{H}(\boldsymbol{R}, t)$ onto S can be written as follows;

$$[\nabla_{\perp} \times \mathbf{z}]E_z + \partial_z[\mathbf{z} \times \mathbf{E}] = -\mu_0\partial_t\mathbf{H}$$

$$[\nabla_{\perp} \times \mathbf{z}]H_z + \partial_z[\mathbf{z} \times \mathbf{H}] = \varepsilon_0\partial_t\mathbf{E}$$

Utilizing the identity

$$\boldsymbol{A} \times \boldsymbol{B} \times \boldsymbol{C} = \boldsymbol{B}(\boldsymbol{A}.\boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A}.\boldsymbol{B})$$

known as triple product expansion or Lagrange's Formula, the vector multiplications of above equations by the unit vector **z** yield;

$$\nabla_{\perp}E_z = \mu_0\partial_t[\boldsymbol{H} \times \boldsymbol{z}] + \partial_z\boldsymbol{E}$$

$$\nabla_{\perp} H_z = \varepsilon_0 \partial_t [\boldsymbol{z} \times \boldsymbol{E}] + \partial_z \boldsymbol{H}.$$

In a similar fashion, the scalar Maxwell's equations can also be rewritten by using the transverse and longitudinal decompositions of the nabla operator, ∇, and of the fields as shown below;

$$\nabla . \mathbf{E}(\mathbf{R}, \mathrm{t}) = [(\nabla_{\perp} + \mathbf{z}\partial_z).(\boldsymbol{E} + \mathbf{z}E_z)] = \nabla_{\perp}.\boldsymbol{E} + \partial_z E_z = 0$$

$$\nabla . \boldsymbol{H}(\boldsymbol{R}, t) = [(\nabla_{\perp} + \mathbf{z}\partial_z).(\boldsymbol{H} + \mathbf{z}H_z)] = \nabla_{\perp}.\boldsymbol{H} + \partial_z H_z = 0.$$

After the decomposition, the Maxwell's equationsgiven in (3.1) and (3.2) can be grouped into two systems of equations: the system just contain E_z ,

$$\nabla_{\perp} E_z = \mu_0 \partial_t [\boldsymbol{H} \times \boldsymbol{z}] + \partial_z \boldsymbol{E} \qquad (3.6a)$$

$$\varepsilon_0 \partial_t E_z = \nabla_{\perp}.[\boldsymbol{H} \times \mathbf{z}] \qquad (3.6b)$$

$$\partial_z E_z = -\nabla_{\perp}.\boldsymbol{E} \qquad (3.6c)$$

with the system just contain H_z component,

$$\nabla_{\perp} H_z = \varepsilon_0 \partial_t [\boldsymbol{z} \times \boldsymbol{E}] + \partial_z \boldsymbol{H} \qquad (3.7a)$$

$$\mu_0 \partial_t H_z = \nabla_{\perp}.[\boldsymbol{z} \times \boldsymbol{E}] \qquad (3.7b)$$

$$\partial_z H_z = -\nabla_{\perp}.\boldsymbol{H} \qquad (3.7c)$$

are obtained in the form of transverse-longitudinal decompositions without any restriction. It can be seen that the equation (3.2) is written in (3.6c) and (3.7c). From the first couple of the boundary conditions (3.3),

$$\boldsymbol{n}.\boldsymbol{H}|_L = 0 \Rightarrow \boldsymbol{n}.(\boldsymbol{H} + \mathbf{z}H_z)|_{\boldsymbol{L}} = 0 \Rightarrow \boldsymbol{n}.\boldsymbol{H}|_L = 0$$

$$\boldsymbol{l}.\boldsymbol{E}|_L = 0 \Rightarrow \boldsymbol{l}.(\boldsymbol{E} + \mathbf{z}E_z)|_{\boldsymbol{L}} = 0 \Rightarrow \boldsymbol{l}.\boldsymbol{E}|_L = 0 \qquad (3.8)$$

$$\boldsymbol{n}.\boldsymbol{H}|_L = 0, \quad \boldsymbol{l}.\boldsymbol{E}|_L = 0$$

can be written. From the last boundary condition in (3.3),

$$\mathbf{z}.\boldsymbol{E}|_L = 0$$

$$\mathbf{z}.(\boldsymbol{E} + \mathbf{z}E_z)|_L = 0 \Rightarrow E_z|_L = 0$$

are obtained. When the condition, $E_z|_L = 0$ is applied to (3.6b) and (3.6c);

$$(\nabla_\perp . [\boldsymbol{H} \times \mathbf{z}])|_L = 0, \qquad (\nabla_\perp . \boldsymbol{E})|_L = 0 \tag{3.9}$$

equalities are obtained.

CHAPTER 4

SOLVING THE PROBLEM

In this chapter, the problem (3.6)-(3.9) has been solved by using the method of separation of variables. For this problem this means that every field component in (3.6)-(3.7) can be written as the product of two factors. One of these factors depends on only transverse coordinates, r and the other one depends on the variables (z,t). We shall also search for the complete sets of transverse electric and transverse magnetic time-domain modes. Transverse modes appear due to the boundary conditions enforced on the wave because of the physical properties of the waveguide. As an example, a radio wave in a hollow waveguide with perfectly conducting surface must have zero-valued tangential electric field component at the walls of the waveguide. Hence, the transverse pattern of electric field is restricted to those which fit between the walls. The allowed modes can be found via solving Maxwell's equations under the boundary conditions over a waveguide surface. These solutions can be divided into two basic sets of modes in the interior of a waveguide. For the one set of modes, there is no axial electric field component. However, these modes have an axial magnetic field and are called ''magnetic type'' that is H modes or ''transverse electric'' that is TE modes. The other basic set of modes has no magnetic field along the waveguide axis but has an axial electric field instead, hence, they are referred to as ''electric type'' that is E modes or ''transverse magnetic'' that is TM modes.

4.1 Complete Set of the Transverse Electric (TE) Time Domain Modes

TE modes are determined by $E_z(\boldsymbol{r}, z, t) = 0$ condition. Substituting this condition in (3.6), $0 = \nabla_{\perp}.[\boldsymbol{H} \times \mathbf{z}]$ and $0 = -\nabla_{\perp}.\boldsymbol{E}$ are obtained from (3.6b) and (3.6c),

$$\nabla_{\perp}.[\boldsymbol{H} \times \mathbf{z}] = -\nabla_{\perp}.[\mathbf{z} \times \boldsymbol{H}] = -[\nabla_{\perp} \times \mathbf{z}].\boldsymbol{H} = 0$$

$$\nabla_{\perp}.\boldsymbol{E} = 0$$

vector identities

$$\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A}, \qquad \boldsymbol{A}.(\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{C}.(\boldsymbol{A} \times \boldsymbol{C}), \qquad \boldsymbol{A}.\boldsymbol{B} = \boldsymbol{B}.\boldsymbol{A}$$

are used in the first line, respetively.

In order to obtain the identity $[\nabla_{\perp} \times \nabla_{\perp}\psi(\boldsymbol{r})] = 0$, thetransverse components of electric and magnetic field strength vectors can be written as;

$$\boldsymbol{E}(\boldsymbol{r}, z, t) = V(z,t)[\nabla_{\perp}\psi(\boldsymbol{r}) \times \boldsymbol{z}]$$

$$\boldsymbol{H}(\boldsymbol{r}, z, t) = I(z,t)\nabla_{\perp}\psi(\boldsymbol{r}). \tag{4.1}$$

The functions $\psi(\boldsymbol{r}), V(z,t)$ and $I(z,t)$ have been found later. According to the system (3.7), H_z have been considered as $H_z = A(z,t)\psi(\boldsymbol{r})$. The unknown function A(z,t) has also been found later. When H_z and the field vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ in (4.1) are substituted in (3.7b) and (3.7c) ,

$$-\mu_0\psi(\boldsymbol{r})\partial_t A(z,t) = -V(z,t)\nabla_{\perp}.[\boldsymbol{z} \times [\nabla_{\perp}\psi(\boldsymbol{r}) \times \boldsymbol{z}]],$$

$$\psi(\boldsymbol{r})\partial_z A(z,t) = -I(z,t)\nabla_{\perp}.\nabla_{\perp}\psi(\boldsymbol{r}),$$

are obtained. After applying the triple product expansion to the right-hand side of the first line, and making use of $\nabla_{\perp}.\nabla_{\perp}\psi(\boldsymbol{r}) = \nabla_{\perp}^2\psi(\boldsymbol{r})$, we obtain

$$-\mu_0\psi(\boldsymbol{r})\partial_t A(z,t) = [-\nabla_{\perp}^2\psi(\boldsymbol{r})]V(z,t)$$

$$\psi(\boldsymbol{r})\partial_z A(z,t) = [-\nabla_{\perp}^2\psi(\boldsymbol{r})]I(z,t)(4.2)$$

which indicates that $\psi(\boldsymbol{r})$ is twice differentiable with respect to the transverse coordinates, $(\boldsymbol{r})$. Lastly, substituting (4.1) and $H_z = A(z,t)\psi(\boldsymbol{r})$ in (3.7a) and using

$\boldsymbol{z} \times [\nabla_{\perp}\psi(\boldsymbol{r}) \times \boldsymbol{z}] = \nabla_{\perp}\psi(\boldsymbol{r})$,and eliminating the term $\nabla_{\perp}\psi(\boldsymbol{r})$ from both sides of the equation,

$$\nabla_{\perp}H_z = \varepsilon_0\partial_t[\boldsymbol{z} \times \boldsymbol{E}] + \partial_z\boldsymbol{H}$$

$$\nabla_{\perp}[A(z,t)\psi(\boldsymbol{r})] = \varepsilon_0\partial_t\left[\boldsymbol{z} \times \left[V(z,t)[\nabla_{\perp}\psi(\boldsymbol{r}) \times \boldsymbol{z}]\right]\right] + \partial_z[I(z,t)\nabla_{\perp}\psi(\boldsymbol{r})]$$

$$A(z,t) = \varepsilon_0\partial_t V(z,t) + \partial_z I(z,t) \tag{4.3}$$

are found. In (3.8), every boundary condition for the transverse components of electric and magnetic field vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ gives the same boundary condition for the function $\psi(\boldsymbol{r})$ as follows; Substitution of (4.1) into (3.8) yields the following conditions;

$$I(z,t)[\boldsymbol{n}.\nabla_{\perp}\psi(\boldsymbol{r})]|_{L} = 0$$

$$V(z,t)(\boldsymbol{l}.[\nabla_{\perp}\psi(\boldsymbol{r}) \times \boldsymbol{z}])|_{L} = 0.$$

Notice that $I(z,t)$ and $V(z,t)$ must not be zero, otherwise, the transverse components of electric and magnetic field vectors in (4.1) will be zero. Since nontrivial solutions are searched for, the above equations should be interpreted as follows;

$$[\boldsymbol{n}.\nabla_{\perp}\psi(\boldsymbol{r})]|_{L} = 0$$

$$(\boldsymbol{l}.[\nabla_{\perp}\psi(\boldsymbol{r}) \times \boldsymbol{z}])|_{L} = 0.$$

Hence, these boundary conditions yield the same condition for the potential $\psi(\boldsymbol{r})$ as follows

$$\partial_{\boldsymbol{n}}\psi(\boldsymbol{r})|_{\boldsymbol{L}} = 0 \tag{4.4}$$

stating that the normal derivative of $\psi(\boldsymbol{r})$ on the contour L equals to zero. Moreover, the definitions in (4.1) satisfy the boundary conditions in (3.9). It is seen from (4.2) that the potential $\psi(\boldsymbol{r})$must be twice differentiable. This situation and the boundary condition (4.4) propose to use Neumann boundary-eigenvalue problem to obtain the TE modal fields. $\boldsymbol{r} \in S$ and $\upsilon_n^2 \geq 0$ $(n = 0,1,2,\dots)$ are the real eigenvalues. For the TE modes, the following problem which is a Neumann boundary- eigenvalue problem is solved.

$$\left(\nabla_{\perp}^2 + \upsilon_n^2\right)\psi_n(\boldsymbol{r}) = 0\,, \quad \partial_{\boldsymbol{n}}\psi_n(\boldsymbol{r})|_{\boldsymbol{L}} = 0 \tag{4.5}$$

where the functions set $\{\psi_n(\boldsymbol{r})\}_{n=0}^{\infty}$ are the eigenfunctions according to the eigenvalues $\{\upsilon_n^2\}_{n=0}^{\infty}$ in problem (4.5) and these functions are complete in Hilbert space $L_2(S)$ space [47]. Thus, any potential $\psi(\boldsymbol{r})$ that satisfy the condition (4.4) can be written in terms of functions $\psi_n(\boldsymbol{r})$. To this aim the functions $\psi_n(\boldsymbol{r})$ should be normalized. The complete set of functions, $\{\psi_n(\boldsymbol{r})\}_{n=0}^{\infty}$, generates a complete set of TE modes in the time domain [6]. In (4.1) and (4.2), the potential ψ can be taken as an eigenfunction ψ_n.

When $\left[-\nabla_{\perp}^2\psi_n(\boldsymbol{r})\right] = \upsilon_n^2\psi_n(\boldsymbol{r})$ is substituted in (4.2) and after introducing $A(z,t)$ as

$A(z,t) = \upsilon_n^2 h_n(z,t)$,it is lso used in (4.2). Then,

$$V(z,t) = -\mu_0 \partial_t h_n(z,t) \quad and \quad I(z,t) = \partial_z h_n(z,t)$$

are found. If these equalities and the expression $A(z,t) = \upsilon_n^2 h_n(z,t)$ are substituted in (4.3), the following differential equation is obtained for the function $h_n(z,t)$

$$\partial_{ct}^2 h_n(z,t) - \partial_z^2 h_n(z,t) + \upsilon_n^2 h_n(z,t) = 0 \tag{4.6}$$

where $\partial_{ct} = \partial / c\partial t$ and $c = 1/\sqrt{\varepsilon_0 \mu_0}$. The differential equation in (4.6) is known as the Klein-Gordon Equation (KGE) in mathematical physics [41] and [42]. After these efforts, all the field components of TE modes can be listed as follows

$$E_{zn}^{TE}(\boldsymbol{r}, z, t) = 0,$$

$$\boldsymbol{E}_n^{TE}(\boldsymbol{r}, z, t) = [-\partial_t h_n(z,t)]\mu_0 \nabla_\perp \psi_n(\boldsymbol{r}) \times \boldsymbol{z},$$

$$\boldsymbol{H}_n^{TE}(\boldsymbol{r}, z, t) = [\partial_z h_n(z,t)]\nabla_\perp \psi_n(\boldsymbol{r}), \tag{4.7}$$

$$H_{zn}^{TE}(\boldsymbol{r}, z, t) = [\upsilon_n h_n(z,t)]\upsilon_n \psi_n(\boldsymbol{r}).$$

Where $n = 1,2, \ldots$ and these equations are called constitutive equations. The expressions in the square brackets in (4.7) are the modal amplitudes in physical sense and the terms, which are the vector functions of transverse coordinates, $(\boldsymbol{r})$, originate the modal basis. To obtain the field components with appropriate physical dimensions, it is necessary to normalize the functions, $\psi_n(\boldsymbol{r})$. The functions $\psi_n(\boldsymbol{r})$, $n \neq 0$ can be normalized as follows.

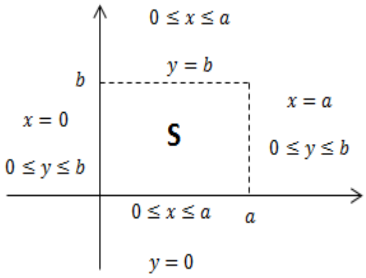
$$\frac{\mu_0 \upsilon_n^2}{S} \int_S |\psi_n(\boldsymbol{r})|^2 ds = 1. \tag{4.8}$$

By using the equalities in (3.5), which enables us to rewrite the energy characteristics of the field in a simpler form as shown below.

$$W_n^{TE}(z,t) = [(\partial_{ct} h_n)^2 + (\partial_z h_n)^2 + \upsilon_n^2 h_n^2]/2$$

$$P_{zn}^{TE}(z,t) = (-\partial_t h_n)(\partial_z h_n).$$

Example 4.1 In this example, a rectangular waveguide will be considered. That is, $0 \leq x \leq a \; and \; 0 \leq y \leq b$ will be considered. The contour of the waveguide will be considered enough properly and none of the inner angle is greater than π.



where $\boldsymbol{r} = (x, y) \in S$. For TE time domain modes, Neumann boundary-eigenvalue problem will be considered.

$$(\nabla_\perp^2 + v_m^2)\psi_m(\boldsymbol{r}) = 0, \qquad \left.\frac{\partial \psi_m(\boldsymbol{r})}{\partial n}\right|_L = 0, \quad \frac{\mu_0 v_m^2}{S}\int_S |\psi_m(\boldsymbol{r})|^2 ds = 1\, N \qquad (4.9)$$

where

$$\partial_n = \boldsymbol{n}.\nabla_\perp$$

is the normal derivative over the L contour and for $m = 1,2,3,\ldots v_m^2 > 0$ are eigenvalues, the potentials $\psi_m(\boldsymbol{r})$ are eigenvectors corresponding to eigenvalues. The solutions of the problem are found using the method of separation of variables.

$$\psi_m(\boldsymbol{r}) \equiv \psi_m(x, y) = X(x)Y(y) \qquad (4.10)$$

$$Y(y)\frac{d^2X(x)}{dx^2} + X(x)\frac{d^2Y(y)}{dy^2} + v_m^2 X(x)Y(y) = 0$$

$$\underbrace{\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}}_{only\ depends\ on\ x;\ -v_x^2} + \underbrace{\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}}_{only\ depends\ on\ y;\ -v_y^2} + v_m^2 = 0$$

So,

$$v_x^2 + v_y^2 = v_m^2$$

can be written. The equation is transformed into a pair of ordinary differential equations.

$$\frac{d^2X(x)}{dx^2} = -X(x)v_x^2 \;\Rightarrow\; X(x) = Acos(v_x x) + Bsin(v_x x)$$

$$\frac{d^2Y(y)}{dy^2} = -Y(y)v_y^2 \;\Rightarrow\; Y(y) = C cos(v_y y) + D sin(v_y y)$$

So, the solution is written as;

$$\psi_m(x,y) = [A cos(v_x x) + B sin(v_x x)][C cos(v_y y) + D sin(v_y y)] \qquad (4.11)$$

Now, the following boundary condition and normalization will be satisfied.

$$\left.\frac{\partial \psi_m(\boldsymbol{r})}{\partial n}\right|_L = 0, \quad \frac{\mu_0 v_m^2}{S}\int_S |\psi_m(\boldsymbol{r})|^2 ds = 1$$

i. For $0 \le x \le a$, $y = 0$, $\boldsymbol{n} = (0,1)$ and

$$\nabla_\perp \psi_m(x,y) = \left((\psi_m)_x, (\psi_m)_y\right)$$

$$\boldsymbol{n}.\nabla_\perp \psi_m = (\psi_m)_y$$

$$(\psi_m)_y(x,0) = [A cos(v_x x) + B sin(v_x x)]D v_y = 0 \;\Rightarrow D = 0$$

Substitute this in (4.10);

$$\psi_m(x,y) = [A cos(v_x x) + B sin(v_x x)][C cos(v_y y)]$$

will be obtained.

ii. For $x = 0$, $0 \le y \le b$, $\boldsymbol{n} = (1,0)$

$$\nabla_\perp \psi_m(x,y) = \left((\psi_m)_x, (\psi_m)_y\right)$$

$$\boldsymbol{n}.\nabla_\perp \psi_m = (\psi_m)_x$$

$$(\psi_m)_x = [-A v_x sin(v_x x) + B v_x cos(v_x x)][C cos(v_y y)]$$

$$(\psi_m)_x(0,y) = B C v_x cos(v_y y) = 0 \;\Rightarrow B = 0$$

İf we rearrange our solution;

$$\psi_m(x,y) = AC cos(v_x x) cos(v_y y)$$

is obtained.

iii. For $x = a, 0 \le y \le b$, $\boldsymbol{n} = (1,0)$

$$\nabla_\perp \psi_m(x,y) = \left((\psi_m)_x, (\psi_m)_y\right)$$

$\boldsymbol{n}.\nabla_{\perp}\psi_m = (\psi_m)_x$

$(\psi_m)_x = [-Av_x sin(v_x x)][Ccos(v_y y)]$

$(\psi_m)_x(a,y) = -ACv_x sin(v_x a)cos(v_y y) = 0 \Rightarrow sin(v_x a) = 0 \Rightarrow v_x a = p\pi$,

$\Rightarrow v_x = \frac{p\pi}{a}$, $\quad p \in \mathbb{Z}$

iv. For $0 \leq x \leq a$, $y = b$, $\boldsymbol{n} = (0,1)$

$\nabla_{\perp}\psi_m(x,y) = ((\psi_m)_x, (\psi_m)_y)$

$\boldsymbol{n}.\nabla_{\perp}\psi_m = (\psi_m)_y$

$(\psi_m)_y = -ACv_y cos(v_x x)sin(v_y y)$

$(\psi_m)_y(x,b) = -ACv_y cos(v_x x)sin(v_y b) = 0 \Rightarrow sin(v_y b) = 0 \Rightarrow v_y b = q\pi$,

$\Rightarrow v_y = \frac{q\pi}{a}$, $\quad q \in \mathbb{Z}$

$v_m^2 = v_x^2 + v_y^2 = \left(\frac{p\pi}{a}\right)^2 + \left(\frac{q\pi}{b}\right)^2$, $p + q \neq 0$

$\psi_m(x,y) = ACcos\left(\frac{p\pi}{a}x\right)cos\left(\frac{q\pi}{b}y\right)$

For $m \neq 0$, $\psi_m(x,y)$ will be normalized as;

$$\frac{\mu_0 v_m^2}{S}\int_S |\psi_m(\boldsymbol{r})|^2 ds = 1$$

$$\frac{\mu_0 v_m^2}{ab}\int_0^a\int_0^b \left|ACcos\left(\frac{p\pi}{a}x\right)cos\left(\frac{q\pi}{b}y\right)\right|^2 dydx = 1$$

$$\frac{\mu_0 v_m^2}{ab}(AC)^2\int_0^a \left|cos\left(\frac{p\pi}{a}x\right)\right|^2 dx\int_0^b \left|cos\left(\frac{q\pi}{b}y\right)\right|^2 dy = 1$$

$$\frac{\mu_0 v_m^2}{ab}(AC)^2\int_0^a \frac{1}{2}\left(cos\left(\frac{2p\pi}{a}x\right)+1\right)dx\int_0^b \frac{1}{2}\left(cos\left(\frac{2q\pi}{b}y\right)+1\right)dy = 1$$

$$\frac{\mu_0 v_m^2}{4ab}(AC)^2\left(\frac{a}{2p\pi}sin\left(\frac{2p\pi}{a}x\right)+x\right)_0^a\left(\frac{b}{2q\pi}sin\left(\frac{2q\pi}{b}y\right)+y\right)_0^b = 1$$

$$\frac{\mu_0 \upsilon_m^2}{4ab}(AC)^2 ab = 1$$

$$(AC)^2 = \frac{4}{\mu_0 \upsilon_m^2}$$

$$A_m^{TE} = AC = \frac{2}{\upsilon_m \sqrt{\mu_0}}$$

The condition of normalization allows us to obtain the normalization constant A_m^{TE}.

So the solution to the Neumann problem (4.9) including the normalization constant

$$A_m^{TE} = \frac{2}{\upsilon_m \sqrt{\mu_0}}.$$

$$\psi_m(\boldsymbol{r}) \equiv \psi_m(x,y) = A_m^{TE} cos\left(\frac{p\pi}{a}x\right) cos\left(\frac{q\pi}{b}y\right)$$

These eigenfunctions correspond to the following eigenvalues;

$$\upsilon_m^2 = \left(\frac{p\pi}{a}\right)^2 + \left(\frac{q\pi}{b}\right)^2 \quad , p \in \mathbb{Z}, q \in \mathbb{Z}, p + q \neq 0.$$

4.2 Complete Set of the Transverse Magnetic (TM) Time-Domain Modes

The TM modes are determined by $H_z(\boldsymbol{r}, z, t) = 0$ condition. Under this condition, the systems (3.6) and (3.7) are solved similar to the TE modes. The TM modes are generated by the solutions of the Dirichlet boundary-eigenvalue problem;

$$\left(\nabla_\perp^2 + \kappa_n^2\right)\phi_n(\boldsymbol{r}) = 0\,, \quad \phi_n(\boldsymbol{r})|_{\boldsymbol{L}} = 0\,, \qquad \frac{\varepsilon_0 \kappa_n^2}{S}\int_S |\phi_n(\boldsymbol{r})|^2 ds = 1\,N. \qquad (4.12)$$

$\kappa_n^2 \geq 0$, $n = 0,1,2,...$ are eigenvalues and the functions $\phi_n(\boldsymbol{r})$ are eigenvectors which correspond to the eigenvalues. The eigenvalues, $\{\kappa_n^2\}_{n=0}^{\infty}$ are real. The complete set of eigenvectors, $\{\phi_n(\boldsymbol{r})\}_{n=1}^{\infty}$ produce a set of complete TM Modes [6], [11], [47]. Components of field for $n = 0,1,2,...$ are;

$$H_{zn}^{TM}(\boldsymbol{r}, z, t) = 0,$$

$$\boldsymbol{H}_n^{TM}(\boldsymbol{r}, z, t) = [-\partial_t e_n(z,t)]\boldsymbol{z} \times \varepsilon_0 \nabla_\perp \phi_n(\boldsymbol{r})\,, \qquad (4.13)$$

$\boldsymbol{E}_n^{TM}(\boldsymbol{r}, z, t) = [\partial_z e_n(z,t)]\nabla_\perp \phi_n(\boldsymbol{r}),$

$E_{zn}^{TM}(\boldsymbol{r}, z, t) = [\kappa_n e_n(z,t)]\kappa_n \phi_n(\boldsymbol{r}).$

The modal amplitudes in square brackets are determined by the functions $e_n(z,t)$ and these functions satisfy the following Klein-Gordon Equation (KGE).

$$\partial_{ct}^2 e_n(z,t) - \partial_z^2 e_n(z,t) + \kappa_n^2 e_n(z,t) = 0 \qquad (4.14)$$

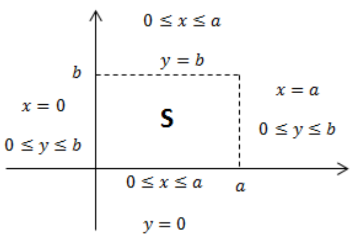
The energy characteristics for TM fields are found by using the normalization in (4.12) as the follow;

$$W_n^{TM}(z,t) = [(\partial_{ct} e_n)^2 + (\partial_z e_n)^2 + \kappa_n^2 e_n^2]/2$$

$$P_{zn}^{TM}(z,t) = (-\partial_t e_n)(\partial_z e_n)$$

Example 4.2 In this example, the Dirichlet problem for a rectangular cross-section will be solved.

$0 \le x \le a\,, \qquad 0 \le y \le b$



$$\left(\nabla_\perp^2 + \kappa_m^2\right)\phi_m(\boldsymbol{r}) = 0, \qquad \phi_m(\boldsymbol{r})|_L = 0, \quad \frac{\varepsilon_0 \kappa_m^2}{S}\int_S |\phi_m(\boldsymbol{r})|^2 ds = 1 \qquad (4.15)$$

The problem is solved by the method of separation of variables.

$$\phi_m(\boldsymbol{r}) \equiv \phi_m(x,y) = X(x)Y(y)$$

$$Y(y)\frac{d^2X(x)}{dx^2} + X(x)\frac{d^2Y(y)}{dy^2} + \kappa_m^2 X(x)Y(y) = 0$$

$$\underbrace{\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}}_{only\ depends\ on\ x;\ -\kappa_x^2} + \underbrace{\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}}_{only\ depends\ on\ y;\ -\kappa_y^2} + \kappa_m^2 = 0$$

where $\kappa_x^2 + \kappa_y^2 = \kappa_m^2$.

The equation is turned into a pair of ordinary differential equations.

$$\frac{d^2X(x)}{dx^2} = -X(x)\kappa_x^2 \;\Rightarrow\; X(x) = Acos(\kappa_x x) + Bsin(\kappa_x x)$$

$$\frac{d^2Y(y)}{dy^2} = -Y(y)\kappa_y^2 \;\Rightarrow\; Y(y) = Ccos(\kappa_y y) + Dsin(\kappa_y y)$$

So, the solution is written as;

$$\phi_m(x,y) = [Acos(\kappa_x x) + Bsin(\kappa_x x)][Ccos(\kappa_y y) + Dsin(\kappa_y y)] \qquad (4.16)$$

Now, let's satisfy the boundary and the normalization conditions;

$$\phi_m(\boldsymbol{r})|_L = 0, \quad \frac{\varepsilon_0 \kappa_m^2}{S}\int_S |\phi_m(\boldsymbol{r})|^2 ds = 1$$

i. For $0 \le x \le a\,, y = 0$;

$$\phi_m(x,0) = [Acos(\kappa_x x) + Bsin(\kappa_x x)]C = 0 \;\Rightarrow\; C = 0$$

Substitute this in (4.16),

$$\phi_m(x,y) = [Acos(\kappa_x x) + Bsin(\kappa_x x)][Dsin(\kappa_y y)]$$

is obtained.

ii. For $x = 0,\; 0 \le y \le b$;

$$\phi_m(0,y) = ADsin(\kappa_y y) = 0 \;\Rightarrow A = 0$$

If we rearrange the solution,

$$\phi_m(x,y) = BDsin(\kappa_x x)sin(\kappa_y y)$$

is obtained.

iii. For $x = a, 0 \le y \le b$;

$$\phi_m(a,y) = BDsin(\kappa_x a)sin(\kappa_y y) = 0$$

$$sin(\kappa_x a) = 0 \;\Rightarrow \kappa_x a = p\pi\,, p \in \mathbb{Z}$$

$$\Rightarrow \kappa_x = \frac{p\pi}{a}\,, \qquad p \in \mathbb{Z}$$

iv. For $0 \le x \le a\,,\ y = b\,,$

$$\phi_m(x,b) = BDsin\left(\frac{p\pi}{a}x\right)sin\left(\kappa_y b\right) = 0$$

$$sin\left(\kappa_y b\right) = 0 \ \ \Rightarrow \kappa_y b = q\pi\,, q \in \mathbb{Z}$$

$$\Rightarrow \kappa_y = \frac{q\pi}{b}\,, \qquad q \in \mathbb{Z}$$

$$\kappa_m^2 = \kappa_x^2 + \kappa_y^2 = \left(\frac{p\pi}{a}\right)^2 + \left(\frac{q\pi}{b}\right)^2\,, \quad p + q \neq 0$$

$$\phi_m(x,y) = BDsin\left(\frac{p\pi}{a}x\right)sin\left(\frac{q\pi}{b}y\right)$$

For $m \neq 0$, let's normalize the function, $\phi_m(x,y)$.

$$\frac{\varepsilon_0 \kappa_m^2}{S}\int_S |\phi_m(\boldsymbol{r})|^2 ds = 1$$

$$\frac{\varepsilon_0 \kappa_m^2}{ab}\int_0^a\int_0^b \left|BDsin\left(\frac{p\pi}{a}x\right)sin\left(\frac{q\pi}{b}y\right)\right|^2 dydx = 1$$

$$\frac{\varepsilon_0 \kappa_m^2}{ab}(BD)^2\int_0^a \left|sin\left(\frac{p\pi}{a}x\right)\right|^2 dx\int_0^b \left|sin\left(\frac{q\pi}{b}y\right)\right|^2 dy = 1$$

$$\frac{\varepsilon_0 \kappa_m^2}{ab}(BD)^2\int_0^a \frac{1}{2}\left(1 - cos\left(\frac{2p\pi}{a}x\right)\right)dx\int_0^b \frac{1}{2}\left(1 - cos\left(\frac{2q\pi}{b}y\right)\right)dy = 1$$

$$\frac{\varepsilon_0 \kappa_m^2}{4ab}(BD)^2\left(x - \frac{a}{2p\pi}sin\left(\frac{2p\pi}{a}x\right)\right)_0^a\left(y - \frac{b}{2q\pi}sin\left(\frac{2q\pi}{b}y\right)\right)_0^b = 1$$

$$\frac{\varepsilon_0 \kappa_m^2}{4ab}(BD)^2 ab = 1$$

$$(BD)^2 = \frac{4}{\varepsilon_0 \kappa_m^2}$$

$$A_m^{TM} = BD = \frac{2}{\kappa_m \sqrt{\varepsilon_0}}$$

Normalization condition allows us to obtain the normalization constant A_m^{TM}. So the solution of the Dirichlet problem (4.15) is obtained as;

$$\phi_m(\boldsymbol{r}) \equiv \phi_m(x,y) = A_m^{TM} sin\left(\frac{p\pi}{a}x\right) sin\left(\frac{q\pi}{b}y\right)$$

where the normalization constant is

$$A_m^{TM} = \frac{2}{\kappa_m \sqrt{\varepsilon_0}} \text{and } p, q \in \mathbb{Z}.$$

The eigenvalues which eigenfunctions correspond to are calculated as follows;

$$\kappa_m^2 = \left(\frac{p\pi}{a}\right)^2 + \left(\frac{q\pi}{b}\right)^2 \quad , p \in \mathbb{Z}, q \in \mathbb{Z}, p + q \neq 0$$

As a general summary;

For TE modal fields;

$$\upsilon_m^2 = \left(\frac{p\pi}{a}\right)^2 + \left(\frac{q\pi}{b}\right)^2 \quad , p \in \mathbb{Z}, q \in \mathbb{Z}, p + q \neq 0$$

and for TM modal fields;

$$\kappa_m^2 = \left(\frac{p\pi}{a}\right)^2 + \left(\frac{q\pi}{b}\right)^2 \quad , p \in \mathbb{Z}, q \in \mathbb{Z}, p + q \neq 0$$

The physical meaning of these eigenvalues is that; the TE and TM waveguide modes which have frequencies smaller than the number $\left(\frac{\mathrm{p}\pi}{\mathrm{a}}\right)^2 + \left(\frac{\mathrm{q}\pi}{\mathrm{b}}\right)^2$ cannot spread in the waveguide. These eigenvalues are called as ‘‘the cut-off frequency’’.

For the TE modes the Klein-Gordon Equation (4.6) and for the TM modes the Klein-Gordon Equation (4.14) have the same structure.

Now, the scaled time τ and the scaled coordinate ξ will be introduced.

For the TE modes, $\quad \tau = \upsilon_m ct\ , \ \xi = \upsilon_m z$

For the TM modes, $\quad \tau = \kappa_m ct\ , \ \xi = \kappa_m z$ (4.17)

So, the Klein –Gordon Equations in (4.6) and (4.14) can be written in a general form as follow.

$$\left(\partial_\tau^2 - \partial_\xi^2 + 1\right) f(\xi,\tau) = 0 \tag{4.18}$$

if $\xi = \upsilon_m z$ and $\tau = \upsilon_m ct$ then $f(\xi,\tau) = h_m(\xi,\tau)$

if $\xi = \kappa_m z$ and $\tau = \kappa_m ct$ then $f(\xi,\tau) = e_m(\xi,\tau)$

Moreover, the TE and the TM modal fields can be rearranged in a common form to study the modal amplitudes of the field components.

$$E_{zn}^{TE}(\boldsymbol{r},\xi,\tau) = 0,$$

$$\upsilon_n^{-1}\sqrt{\varepsilon_0/\mu_0}\,\boldsymbol{E}_n^{TE}(\boldsymbol{r},\xi,\tau) = \mathcal{A}(\xi,\tau)\nabla_\perp\psi_n(\boldsymbol{r}) \times \boldsymbol{z},$$

$$\upsilon_n^{-1}\boldsymbol{H}_n^{TE}(\boldsymbol{r},\xi,\tau) = \mathfrak{B}(\xi,\tau)\nabla_\perp\psi_n(\boldsymbol{r}),$$

$$\upsilon_n^{-1}H_{zn}^{TE}(\boldsymbol{r},\xi,\tau) = f(\xi,\tau)\upsilon_n\psi_n(\boldsymbol{r}),$$

$$H_{zn}^{TM}(\boldsymbol{r},\xi,\tau) = 0,$$

$$\kappa_n^{-1}\sqrt{\mu_0/\varepsilon_0}\,\boldsymbol{H}_n^{TM}(\boldsymbol{r},\xi,\tau) = \mathcal{A}(\xi,\tau)\boldsymbol{z} \times \nabla_\perp\phi_n(\boldsymbol{r}),$$

$$\kappa_n^{-1}\boldsymbol{E}_n^{TM}(\boldsymbol{r},\xi,\tau) = \mathfrak{B}(\xi,\tau)\nabla_\perp\phi_n(\boldsymbol{r}),$$

$$\kappa_n^{-1}E_{zn}^{TM}(\boldsymbol{r},\xi,\tau) = f(\xi,\tau)\kappa_n\phi_n(\boldsymbol{r}).$$

where $\mathcal{A}(\xi,\tau)$ and $\mathfrak{B}(\xi,\tau)$ are the amplitudes of the transverse field components. They are generated by the function $f(\xi,\tau)$, which is the amplitude of the longitudinal field component. They are

$\mathcal{A}(\xi,\tau) = -\partial_\tau f(\xi,\tau)$ and $\mathfrak{B}(\xi,\tau) = \partial_\xi f(\xi,\tau)$.

Therefore, for the TE and the TM modal fields, the energy density and the z-component of Poynting vector are written in the new forms as follows:

$$k_n^{-2}W(\xi,\tau) = (\mathcal{A}^2 + \mathfrak{B}^2 + f^2)/2$$

$$k_n^{-2}P_z(\xi,\tau) = c\mathcal{A}\mathfrak{B}$$

In this case, the conservation of energy can be interpreted as,

$$\partial_\xi P_z(\xi,\tau) + c\partial_\tau W(\xi,\tau) = 0$$

in the differential form.

4.3 Lorentz Invariance of Klein-Gordon Equation

As the basis of the theory of relativity, it is claimed that all the laws of physics are described in the same form by observers moving at constant speed relative to each other. In this part,it is shown that the Klein-Gordon Equation is invariant under the Lorentz transformation which is modified form of the Newton and the Galilean transformations. Using Lorentz transformation, space and time measurements for two different observers can be converted to each other in their own frame of reference.

$\bar{\gamma} = 1/\sqrt{1-\beta^2}$ is Lorentz factor and $\beta = v_z/c$. The Lorentz transformation is defined as;

$$x = \bar{x}, \qquad y = \bar{y},$$

$$z = \bar{\gamma}(\bar{z} + v_z\bar{t}), \quad t = \bar{\gamma}\left(\bar{t} + \frac{v_z}{c^2}\bar{z}\right) \tag{4.19}$$

The inverse Lorentz transformation is also given as,

$$\bar{x} = x, \qquad \bar{y} = y,$$

$$\bar{z} = \bar{\gamma}(z - v_z t), \qquad \bar{t} = \bar{\gamma}\left(t - \frac{v_z}{c^2}z\right). \tag{4.20}$$

The Klein-Gordon Equation (4.18) has the same form in both the reference frame and the moving frame. To prove this, the second line of (4.19) is rearranged by using (4.17). It is known from (4.17) that; $\tau = \upsilon_m ct$, $\xi = \upsilon_m z$ and $\tau = \kappa_m ct$, $\xi = \kappa_m z$. When z and t which are in (4.19) are substituted here,

$$\xi = \upsilon_m\{\bar{\gamma}(\bar{z} + v_z\bar{t})\} \Rightarrow \xi = \bar{\gamma}\left(\bar{\xi} + \beta\bar{\tau}\right)$$

$$\tau = \upsilon_m c\left\{\bar{\gamma}\left(\bar{t} + \frac{v_z}{c^2}\bar{z}\right)\right\} \Rightarrow \tau = \bar{\gamma}\left(\bar{\tau} + \beta\bar{\xi}\right) \tag{4.21}$$

and similarly,

$$\bar{\xi} = \bar{\gamma}(\xi - \beta\tau)$$

$\bar{\tau} = \bar{\gamma}(\tau - \beta\xi)$ (4.22)

are obtained.

In this case, the solution of the equation (4.18) can be interpreted as;

$f[\bar{\xi}(\xi,\tau); \bar{\tau}(\xi,\tau)]$.

Taking partial derivatives with respect to ξ and τ,

$\partial_\tau f(\bar{\xi},\bar{\tau}) = \bar{\gamma}(-\beta\partial_{\bar{\xi}} + \partial_{\bar{\tau}})f(\bar{\xi},\bar{\tau})$,

$\partial_\xi f(\bar{\xi},\bar{\tau}) = \bar{\gamma}(\partial_{\bar{\xi}} - \beta\partial_{\bar{\tau}})f(\bar{\xi},\bar{\tau})$

are obtained.

Taking advantage of the first order partial derivatives, the second order partial derivatives are also calculated as follows,

$\partial_\tau^2 f(\bar{\xi},\bar{\tau}) = \left(\beta^2\bar{\gamma}^2\partial_{\bar{\xi}}^2 - 2\beta\bar{\gamma}^2\partial_{\bar{\xi}}\partial_{\bar{\tau}} + \bar{\gamma}^2\partial_{\bar{\tau}}^2\right)f(\bar{\xi},\bar{\tau})$,

$\partial_\xi^2 f(\bar{\xi},\bar{\tau}) = \left(\bar{\gamma}^2\partial_{\bar{\xi}}^2 - 2\beta\bar{\gamma}^2\partial_{\bar{\xi}}\partial_{\bar{\tau}} + \beta^2\bar{\gamma}^2\partial_{\bar{\tau}}^2\right)f(\bar{\xi},\bar{\tau})$.

Substituting the second order partial derivatives in equation (4.18), it is seen that the Klein-Gordon Equation (KGE) also has the same form in the moving frame as follows;

$\left(\partial_{\bar{\tau}}^2 - \partial_{\bar{\xi}}^2 + 1\right)f(\bar{\xi},\bar{\tau}) = 0$.

4.4 The Solution of Klein-Gordon Equation (KGE)

Willard Miller has formed eleven invertable functions which are suitable for the Klein-Gordon Equation. The results are very crucial for developing of electromagnetic theory in the time domain. According to Miller's idea, the solution $f(\xi,\tau)$ to the KGE in (4.18) can be interpreted as a function of new independent variables $u(\xi,\tau)$ and $v(\xi,\tau)$. It is supposed that $u(\xi,\tau)$ and $v(\xi,\tau)$ are twice differentiable functions of the old variables (ξ,τ) .Substitution of $f[u(\xi,\tau), v(\xi,\tau)]$ into (4.18) the following equation;

$(\partial_\tau^2 - \partial_\xi^2 + 1)f(u(\xi,\tau), v(\xi,\tau)) = 0$

is obtained. After computing the partial derivatives as follows, they will be substituted in the above equation.

$$\partial_\tau f(u,v) = \partial_u f.\partial_\tau u + \partial_v f.\partial_\tau v$$

$$\partial_\xi f(u,v) = \partial_u f.\partial_\xi u + \partial_v f.\partial_\xi v$$

$$\partial_\tau{}^2 f(u,v) = \partial_u{}^2 f.(\partial_\tau u)^2 + \partial_u f.\partial_\tau{}^2 u + \partial_v{}^2 f.(\partial_\tau v)^2 + \partial_v f.\partial_\tau{}^2 v + \partial_{uv}{}^2 f.\partial_\tau u.\partial_\tau v + \partial_{vu}{}^2 f.\partial_\tau v.\partial_\tau u$$

This last equation can be simplified as follow because u and v are independent variables.

$$\partial_\tau{}^2 f(u,v) = \partial_u{}^2 f.(\partial_\tau u)^2 + \partial_v{}^2 f.(\partial_\tau v)^2 + \partial_u f.\partial_\tau{}^2 u + \partial_v f.\partial_\tau{}^2 v + 2\partial_{uv}{}^2 f.\partial_\tau u.\partial_\tau v$$

And similarly,

$$\partial_\xi^2 f(u,v) = \partial_u{}^2 f.(\partial_\xi u)^2 + \partial_v{}^2 f.(\partial_\xi v)^2 + \partial_u f.\partial_\xi{}^2 u + \partial_v f.\partial_\xi{}^2 v + 2\partial_{uv}{}^2 f.\partial_\xi u.\partial_\xi v$$

After substitution of the derivatives into their places,

$$\left\{\left[(\partial_\tau u)^2 - \left(\partial_\xi u\right)^2\right]\partial_u^2 + \left[(\partial_\tau v)^2 - \left(\partial_\xi v\right)^2\right]\partial_v^2 + \left[\partial_\tau^2 u - \partial_\xi^2 u\right]\partial_u + \left[\partial_\tau^2 v - \partial_\xi^2 v\right]\partial_v + 2\left[(\partial_\tau v)(\partial_\tau u) - \left(\partial_\xi v\right)\left(\partial_\xi u\right)\right]\partial_{uv}^2 + 1\right\} f(u,v) = 0 \qquad (4.23)$$

is obtained.

4.5 The Modal Amplitudes in the Form of the Product of Airy Functions

In Miller's list, the fifth pair of the equations is considered, where

$-\infty < u\,, v < \infty;$

$u + v = (\tau + \xi)/2$

$u - v = \pm\sqrt{\tau - \xi}$; given (-) rather than (+)

$$u = \frac{\tau + \xi}{4} - \frac{\sqrt{\tau - \xi}}{2}$$

$$v = \frac{\tau + \xi}{4} + \frac{\sqrt{\tau - \xi}}{2}$$

After computing the partial derivatives as follows,

$$\partial_\tau u = \frac{1}{4} - \frac{1}{4}(\tau - \xi)^{\frac{-1}{2}}$$

$$\partial_\tau^2 u = \frac{1}{8}(\tau - \xi)^{\frac{-3}{2}}$$

$$\partial_\xi u = \frac{1}{4} + \frac{1}{4}(\tau - \xi)^{\frac{-1}{2}}$$

$$\partial_\xi^2 u = \frac{1}{8}(\tau - \xi)^{\frac{-3}{2}}$$

$$\partial_\tau v = \frac{1}{4} + \frac{1}{4}(\tau - \xi)^{\frac{-1}{2}}$$

$$\partial_\tau^2 v = -\frac{1}{8}(\tau - \xi)^{\frac{-3}{2}}$$

$$\partial_\xi v = \frac{1}{4} - \frac{1}{4}(\tau - \xi)^{\frac{-1}{2}}$$

$$\partial_\xi^2 v = -\frac{1}{8}(\tau - \xi)^{\frac{-3}{2}}$$

$$\partial_\xi u.\partial_\xi v = [\frac{1}{4} + \frac{1}{4}(\tau - \xi)^{\frac{-1}{2}}][\frac{1}{4} - \frac{1}{4}(\tau - \xi)^{\frac{-1}{2}}]$$

$$\partial_\tau u.\partial_\tau v = [\frac{1}{4} - \frac{1}{4}(\tau - \xi)^{\frac{-1}{2}}][\frac{1}{4} + \frac{1}{4}(\tau - \xi)^{\frac{-1}{2}}]$$

they will be substituted into equation (4.23) and it is seen that the coefficients in the square brackets of (4.23) are obtained as follows

$$\left[(\partial_\tau u)^2 - \left(\partial_\xi u\right)^2\right] = -\left[(\partial_\tau v)^2 - \left(\partial_\xi v\right)^2\right] = \frac{1}{4(u - v)}$$

and all the other coefficients are equal to zero, so it is obtained that

$$\frac{\partial^2 f(u,v)}{\partial u^2} + 4uf(u,v) = \frac{\partial^2 f(u,v)}{\partial v^2} + 4vf(u,v)$$

Using the method of separation of variables gives

$$f(u,v) = U(u).V(v)$$

$$\frac{1}{U(u)}\frac{d^2}{du^2}U(u) + 4u = \frac{1}{V(v)}\frac{d^2}{dv^2}V(v) + 4v = 4\alpha$$

where α is a constant of separation of the variables. At this point, we introduce two new auxiliary variables $\bar{u}$ and $\bar{v}$ as

$$\bar{u} = \sqrt[3]{4}(\alpha - u)$$

$$\bar{v} = \sqrt[3]{4}(\alpha - v)$$

Then by using the auxiliary variables in the above equations, the following Airy differential equation

$$\frac{d^2}{d\bar{u}^2}U(\bar{u}) - \bar{u}U(\bar{u}) = 0 \quad and \quad \frac{d^2}{d\bar{v}^2}V(\bar{v}) - \bar{v}V(\bar{v}) = 0 \tag{4.24}$$

are obtained [23], [44]. The Airy functions as the solutions to (4.24) are real valued, provided that the constant parameter α in their arguments given in the auxiliary variables above is chosen to be real valued. The final solution to the Klein-Gordon equation can be written as follow.

$$f(\xi,\tau) = U(\bar{u})V(\bar{v})\,, \quad 0 \le \xi \le \tau$$

Assume that the solution has a beginning at t=0. Physically, this condition means that the field sources are not active before t=0.

In accordance with the causality principle the solution $f(\xi,\tau)$ to the Klein-Gordon equation must be expressed in the following piece-wise function.

$$f(\xi,\tau) = \begin{cases} 0 & , \quad \tau < 0 \\ U(\bar{u})V(\bar{v}) & , \quad 0 \le \xi \le \tau \\ 0 & , \quad \xi > \tau \end{cases}$$

The first line of the function above is named as ‘‘the weak causality principle’’. For $t < 0$, when the field sources are zero, all fields in that time is also zero. The last line is the strong causality condition [43]. Moreover, according to the axiom of the special theory of relativity, an electromagnetic field transmits signal with the velocity of light, c. These last two conditions together means that the modal fields, created by a source activated at $t = 0$, are zero beyond the distance $z = ct$, i.e. for $z > ct$. The solution

$f(\xi,\tau) = U(\bar{u})V(\bar{v})$ assumes implicitly that the initial conditions of (2.86) are chosen as

$$f(\xi,\tau)|_{\xi=0} = U(\bar{u})V(\bar{v})|_{\xi=0} = \begin{cases} \varphi(\tau), & \tau \geq 0 \quad \Rightarrow for\ t \geq 0 \\ 0, & \tau < 0 \quad \Rightarrow for\ t < 0 \end{cases}$$

$$-\frac{\partial}{\partial\tau}f(\xi,\tau)\bigg|_{\xi=0} = -\frac{\partial}{\partial\tau}[U(\bar{u})V(\bar{v})]\bigg|_{\xi=0} = \begin{cases} \hat{\varphi}(\tau), & \tau \geq 0 \quad \Rightarrow for\ t \geq 0 \\ 0, & \tau < 0 \quad \Rightarrow for\ t < 0 \end{cases}.$$

Hence, our procedure has sense of the expansion of initial values, given for $\xi = 0$, over the whole domain of propagation, $0 \leq \xi \leq \tau < +\infty$. Every equation in (4.24) has two linearly independent solutions.

$$U(\bar{u}) = a_1 A_i(\bar{u}) + b_1 B_i(\bar{u})$$

$$V(\bar{v}) = a_2 A_i(\bar{v}) + b_2 B_i(\bar{v})$$

where a_1, a_2, b_1, b_2 are arbitrary constants. $A_i(.)$ is the Airy function of the first kind and $B_i(.)$ is the Airy function of the second kind. Their arguments as the functions of dimensionless time τ and dimensionless coordinate ξ are

$$\bar{u} = \sqrt[3]{4}\left[\alpha - \left(\frac{\tau+\xi}{4} - \frac{\sqrt{\tau-\xi}}{2}\right)\right] \quad and \quad \bar{v} = \sqrt[3]{4}\left[\alpha - \left(\frac{\tau+\xi}{4} + \frac{\sqrt{\tau-\xi}}{2}\right)\right]$$

To have oscillation in Airy functions these arguments should be negative. So, it should be a restriction on α as;

$$\alpha - \left(\frac{\tau+\xi}{4} - \frac{\sqrt{\tau-\xi}}{2}\right) < 0 \quad and \quad \alpha - \left(\frac{\tau+\xi}{4} + \frac{\sqrt{\tau-\xi}}{2}\right) < 0$$

$$\alpha < \left(\frac{\tau+\xi}{4} - \frac{\sqrt{\tau-\xi}}{2}\right) \quad and \quad \alpha < \left(\frac{\tau+\xi}{4} + \frac{\sqrt{\tau-\xi}}{2}\right)$$

From the characteristic line of KGE,

$$\left(\partial_\tau^2 - \partial_\xi^2 + 1\right)f(\xi,\tau) = 0.$$

$$y = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}x + c \quad where\ A = -1, \quad B = 0, \quad C = 0,$$

c is constant. $\tau = -\xi + c \quad and \quad \tau = \xi + c$ will be found. For $\tau = \xi + c \ \ , c < 1$

Let's choose $c = 0.05$ and for the smaller τ and ξ , $\tau = \xi + 0.05$ and $0 \leq \xi \leq 1$, so an upper bound for α can be found as

$$\alpha < \frac{\tau + \xi}{4} - \frac{\sqrt{\tau - \xi}}{2} = \frac{0.05}{4} - \frac{\sqrt{0.05}}{2} = -0.0930$$

Considering $f(\xi,\tau) = U(\bar{u})V(\bar{v})$ and

$$U(\bar{u}) = a_1 A_i(\bar{u}) + b_1 B_i(\bar{u})$$

$$V(\bar{v}) = a_2 A_i(\bar{v}) + b_2 B_i(\bar{v})$$

together in the domain of propagation, $0 \leq \xi \leq \tau < +\infty$, it is concluded that all the following combinations of the products of Airy functions

$$f_1(\xi,\tau) = A_i(\bar{u})A_i(\bar{v}) \quad , \quad f_2(\xi,\tau) = A_i(\bar{u})B_i(\bar{v})$$

$$f_3(\xi,\tau) = B_i(\bar{u})A_i(\bar{v}) \quad , \quad f_4(\xi,\tau) = B_i(\bar{u})B_i(\bar{v})$$

can have physical sense as the solution $f(\xi,\tau)$, provided that the value of α is chosen properly.

4.6 Surplus of Energy

The analytical method used is based on solving the Maxwell's equations with time derivative which can be expressed in transverse and longitudinal coordinates. The basis functions producing the electric and magnetic fields and the functions transferring signal are found by this method. An electromagnetic wave radiating in space also carries energy with itself. Because of this, the energetic properties of the electric and magnetic fields are also investigated. The energy is dependent on the time, so the wave form of the energy exchange between certain components of the modal fields must also act together with electric-magnetic field propagation. Therefore, the scalar functions transferring signal which are solutions to the Klein-Gordon equation are shown graphically with the energy they carry. In this study, energy and the energy exchange (surplus of energy) are expressed as follows;

$$W(\xi,\tau) = \frac{[\mathcal{A}^2(\xi,\tau) + \mathfrak{B}^2(\xi,\tau) + f^2(\xi,\tau)]}{2} \quad and$$

$$sW(\xi,\tau) = \frac{[\mathcal{A}^2(\xi,\tau) - \mathfrak{B}^2(\xi,\tau)]}{2} \;\; where \;\; \mathcal{A} = -\frac{\partial}{\partial\tau} f(\xi,\tau), \mathfrak{B} = \frac{\partial}{\partial\xi} f(\xi,\tau)$$

where W, sW are energy and energy exchange (surplus of energy), respectively and f is the generating function which is derived from Klein-Gordon equation. $\mathcal{A}$ and $\mathfrak{B}$ are named as the time derivative and the axial coordinate derivative, respectively. The difference between the square of the time derivative and the square of the axial coordinate derivative gives the reader the energy exchange (surplus of energy) in the interval 0 and t. The generating function f, is formed via Airy functions by using a special choice from Miller's list. The function exhibits all the characters of the special function Airy function. It converges for all real arguments. It has been continued to calculate the energy and the energy exchange (surplus of energy). As a result of propagation of the signal function which is obtained analytically, the energy exchange (surplus of energy) occurs. These energy exchanges are shown graphically below by numerical examples. The graphs are time-dependent. When these calculations are made, they are subjected to Einstein's second postulate, causality principle. Graphs of results are encoded in maple programme.

4.7 Time-Dependent Graphics

The energy stored in the transverse and longitudinal components of the modal fields changes alternately when signal transferring is occuring. Surplus of energy is the difference between the stored energy. This is seen in the graphics in detail. The following graphics are obtained by using maple programme.

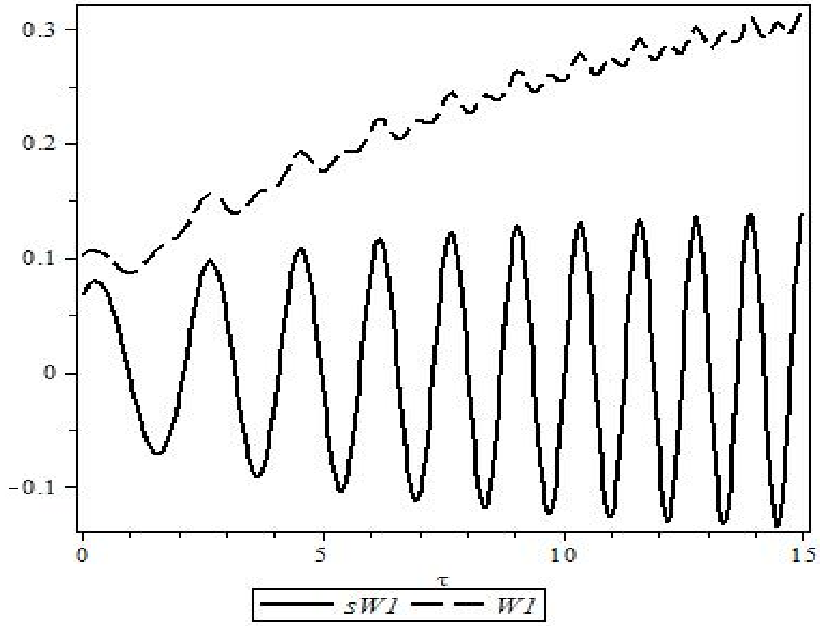


Figure 4.1 The energy W_1 and surplus of energy sW_1 for the source function $f_1(\xi,\tau)$ when $\alpha = -1$ $(0 \leq \tau \leq 15\ )$, ξ is fixed.

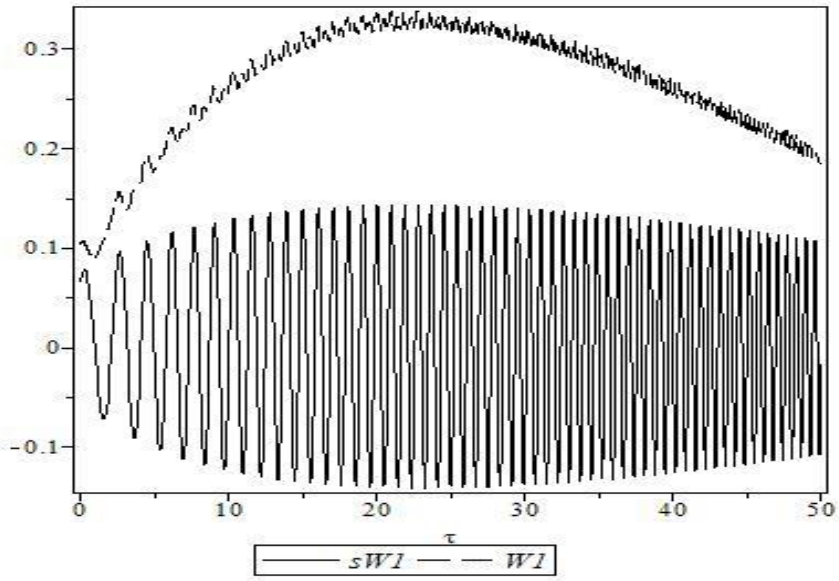


Figure 4.2 The energy W_1 and surplus of energy sW_1 for the source function $f_1(\xi,\tau)$ when $\alpha = -1$ $(0 \leq \tau \leq 50)$, ξ is fixed.

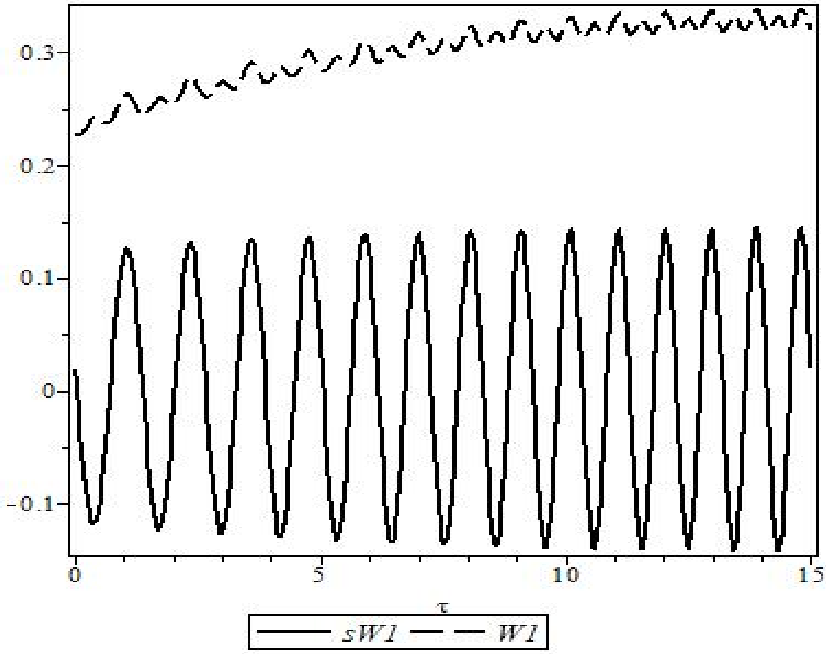


Figure 4.3 The energy W_1 and surplus of energy sW_1 for the source function $f_1(\xi,\tau)$ when $\alpha = -5$ $(0 \leq \tau \leq 15)$, ξ is fixed.

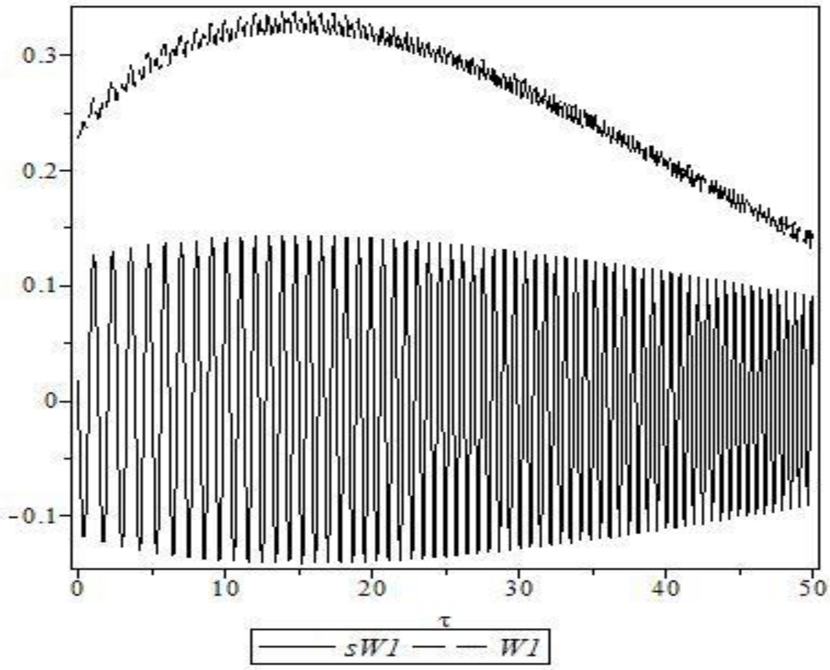


Figure 4.4 The energy W_1 and surplus of energy sW_1 for the source function $f_1(\xi,\tau)$ when $\alpha = -5$ $(0 \leq \tau \leq 50)$, ξ is fixed.

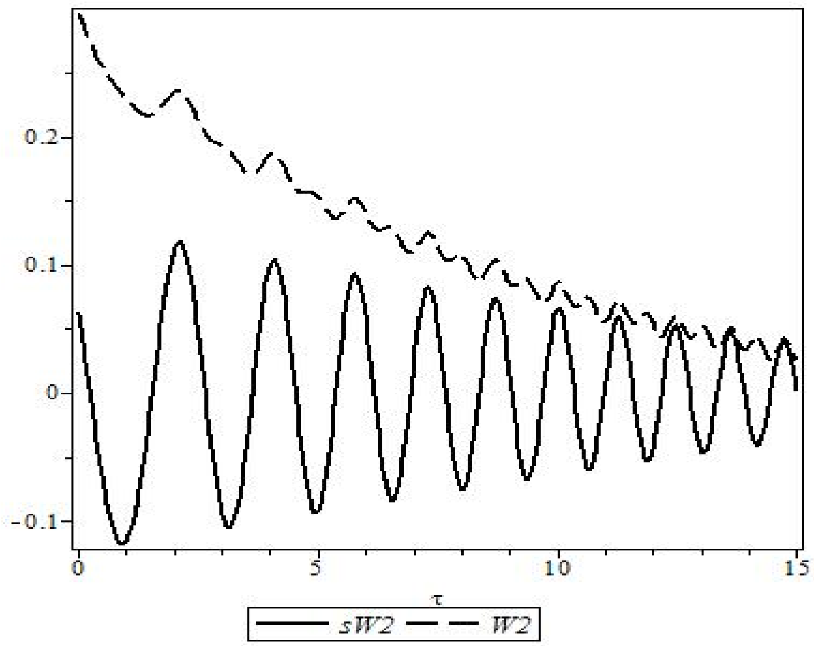


Figure 4.5 The energy W_2 and surplus of energy sW_2 for the source function $f_2(\xi, \tau)$ when $\alpha = -1$ ($0 \leq \tau \leq 15$), ξ is fixed.

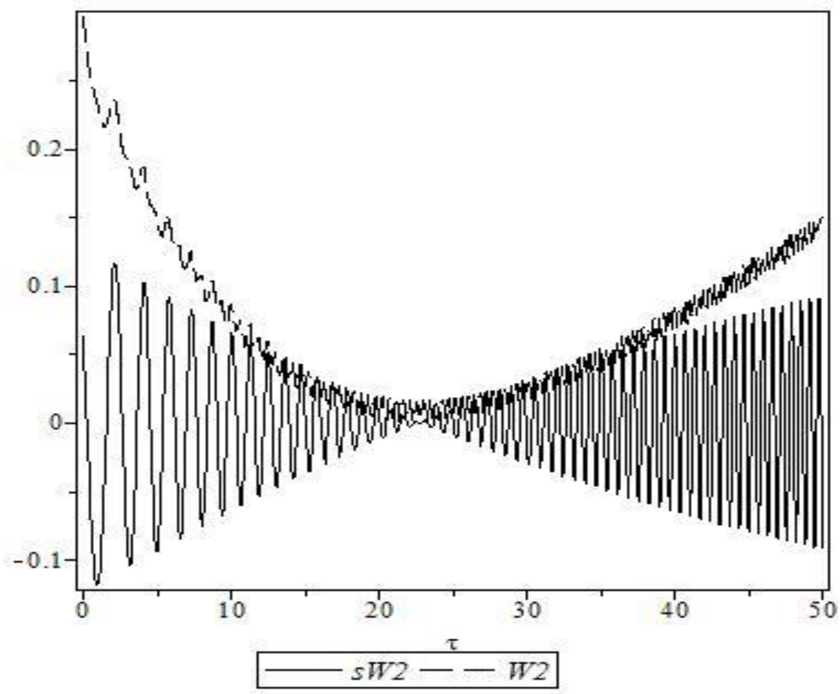


Figure 4.6 The energy W_2 and surplus of energy sW_2 for the source function $f_2(\xi, \tau)$ when $\alpha = -1$ ($0 \leq \tau \leq 50$), ξ is fixed.

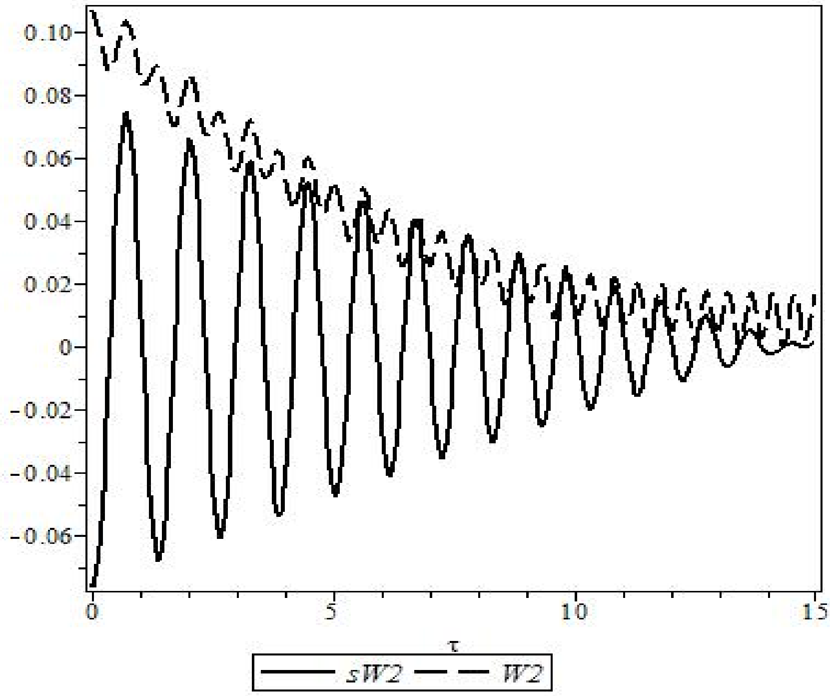


Figure 4.7 The energy W_2 and surplus of energy sW_2 for the source function $f_2(\xi, \tau)$ when $\alpha = -5$ ($0 \leq \tau \leq 15$), ξ is fixed.

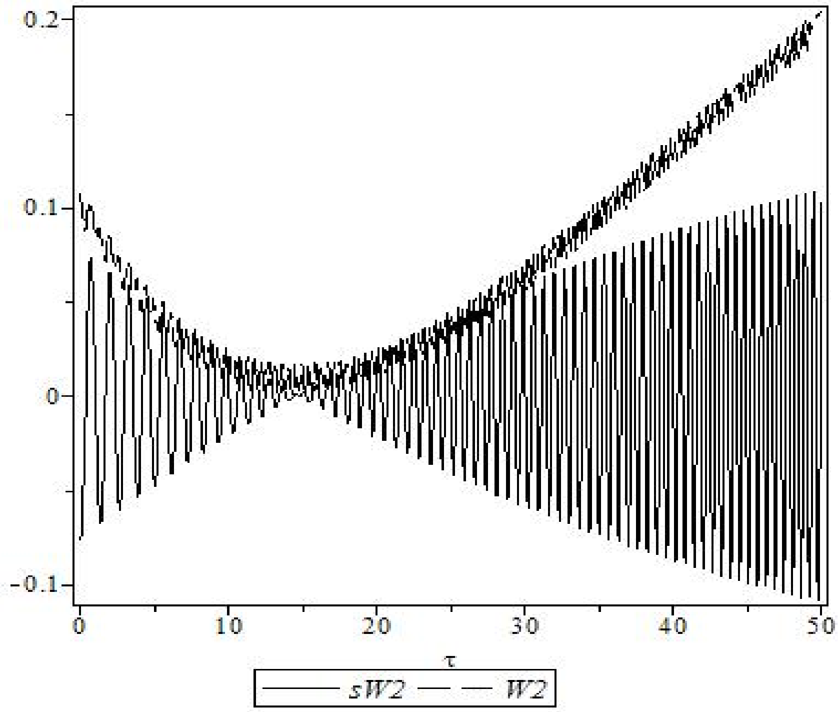


Figure 4.8 The energy W_2 and surplus of energy sW_2 for the source function $f_2(\xi, \tau)$ when $\alpha = -5$ ($0 \leq \tau \leq 50$), ξ is fixed.

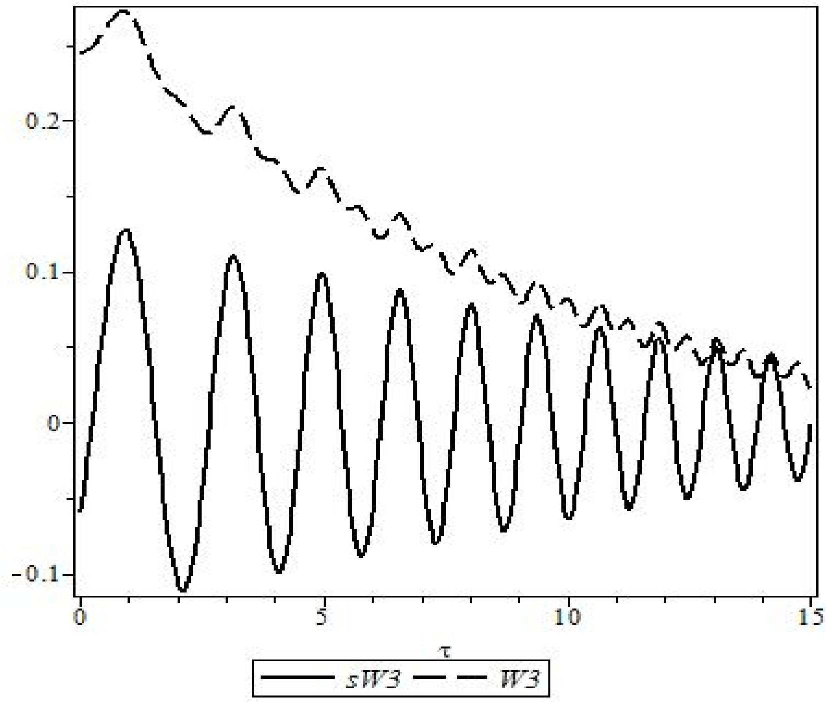


Figure 4.9 The energy W_3 and surplus of energy sW_3 for the source function $f_3(\xi,\tau)$ when $\alpha = -1$ $(0 \leq \tau \leq 15)$, ξ is fixed.

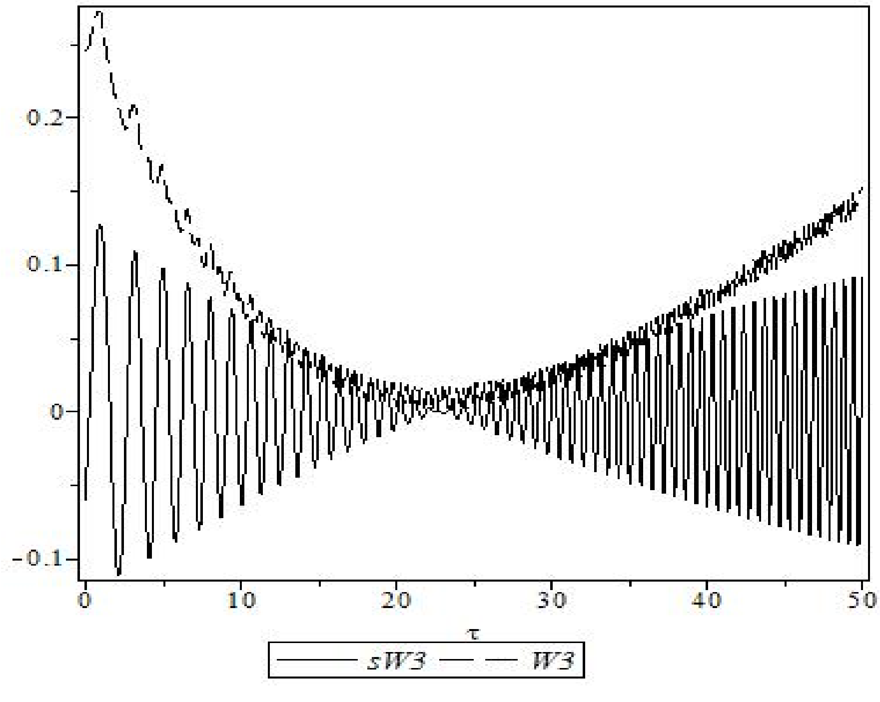


Figure 4.10 The energy W_3 and surplus of energy sW_3 for the source function $f_3(\xi,\tau)$ when $\alpha = -1$ $(0 \leq \tau \leq 50)$, ξ is fixed.

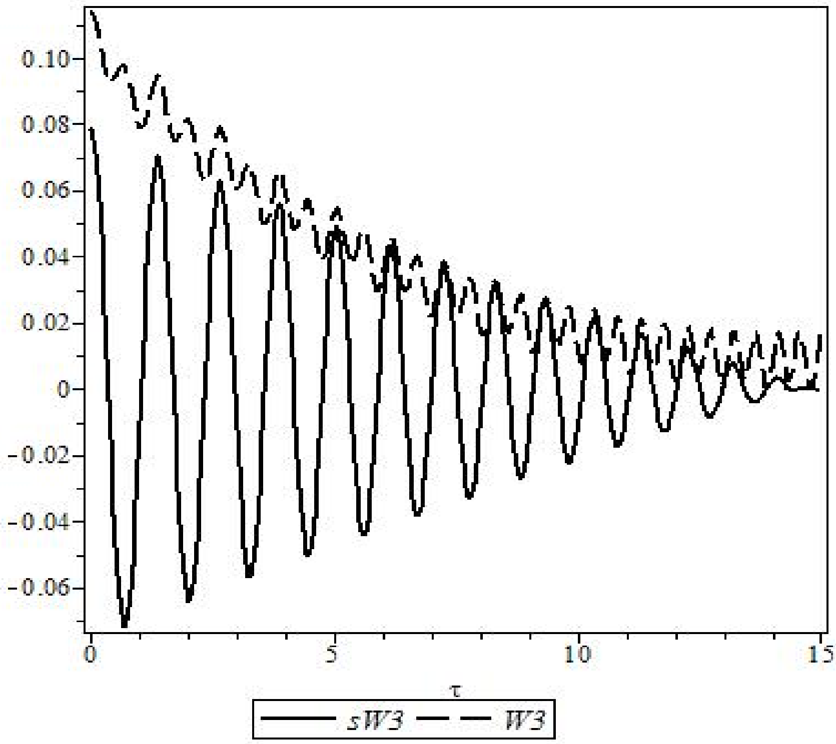


Figure 4.11 The energy W_3 and surplus of energy sW_3 for the source function $f_3(\xi,\tau)$ when $\alpha = -5$ $(0 \le \tau \le 15)$, ξ is fixed.

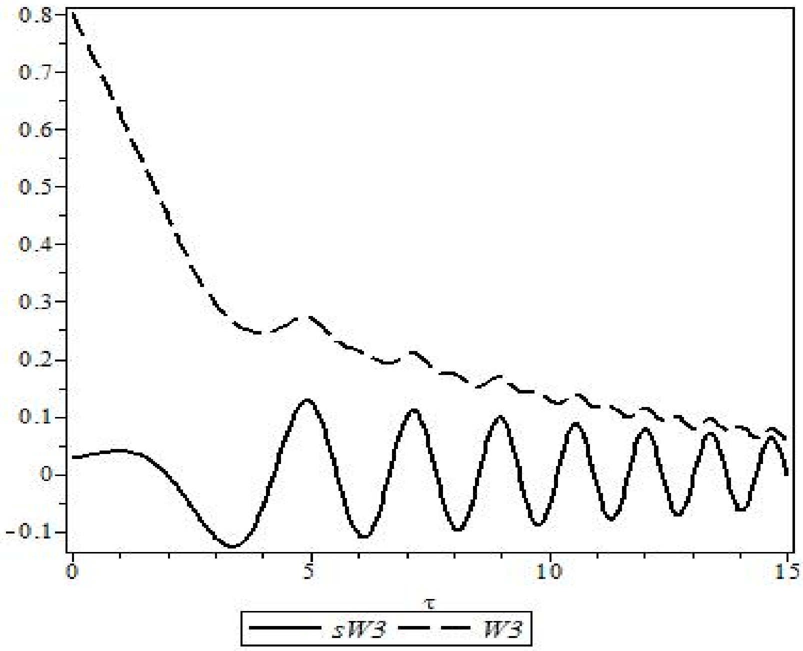


Figure 4.12 The energy W_3 and surplus of energy sW_3 for the source function $f_3(\xi,\tau)$ when $\alpha = -5$ $(0 \le \tau \le 50)$, ξ is fixed.

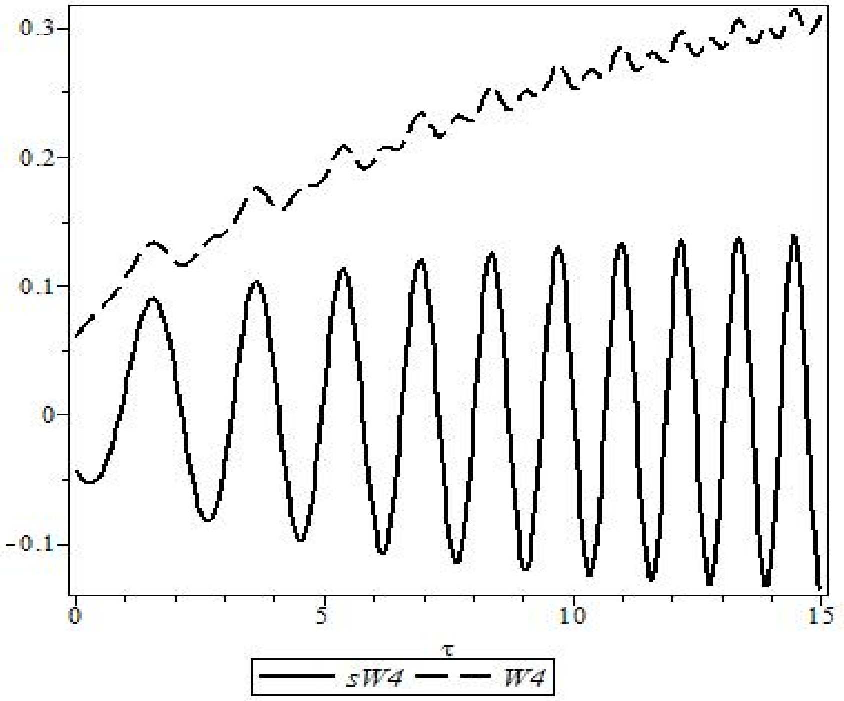


Figure 4.13 The energy W_4 and surplus of energy sW_4 for the source function $f_4(\xi, \tau)$ when $\alpha = -1$ $(0 \leq \tau \leq 15)$, ξ is fixed.

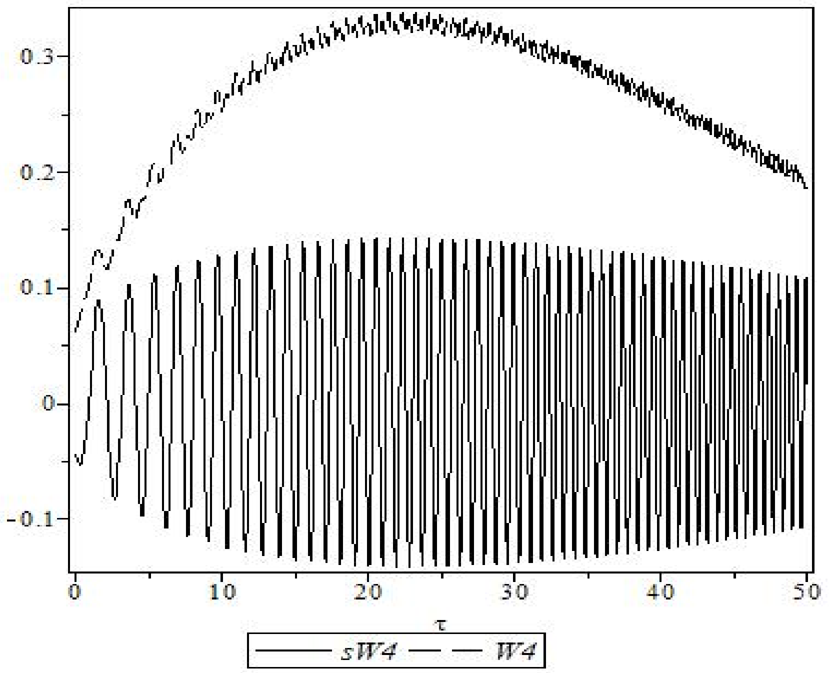


Figure 4.14 The energy W_4 and surplus of energy sW_4 for the source function $f_4(\xi, \tau)$ when $\alpha = -1$ $(0 \leq \tau \leq 50)$, ξ is fixed.

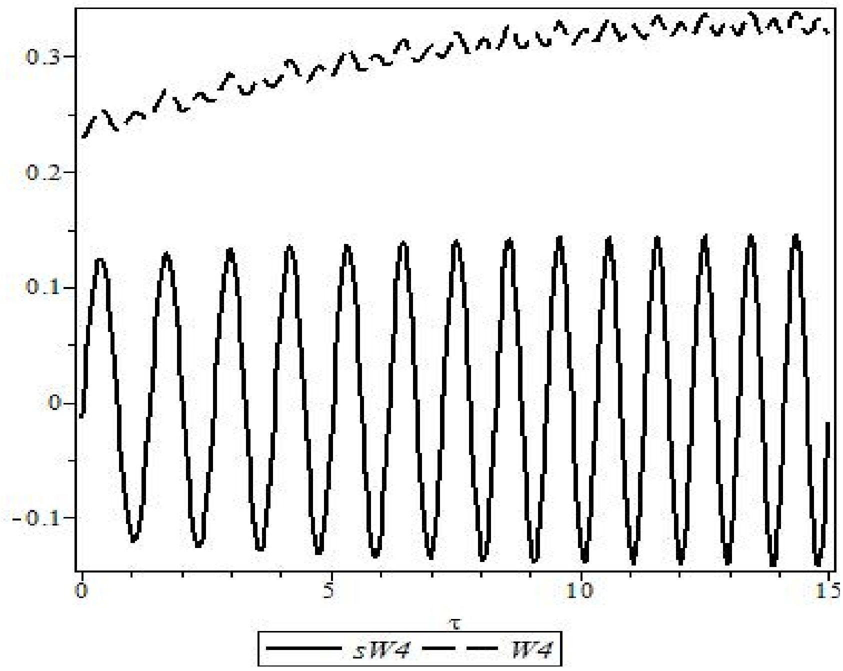


Figure 4.15 The energy W_4 and surplus of energy sW_4 for the source function $f_4(\xi,\tau)$ when $\alpha = -5$ $(0 \le \tau \le 15)$, ξ is fixed.

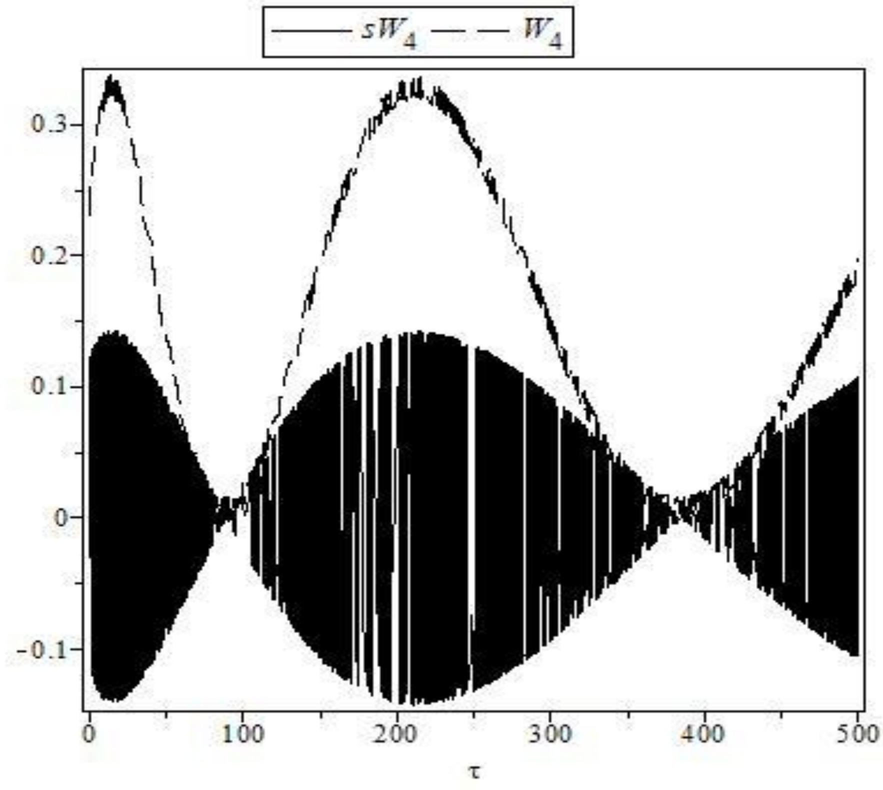


Figure 4.16 The energy W_4 and surplus of energy sW_4 for the source function $f_4(\xi,\tau)$ when $\alpha = -1$ $(0 \le \tau \le 500)$, ξ is fixed.

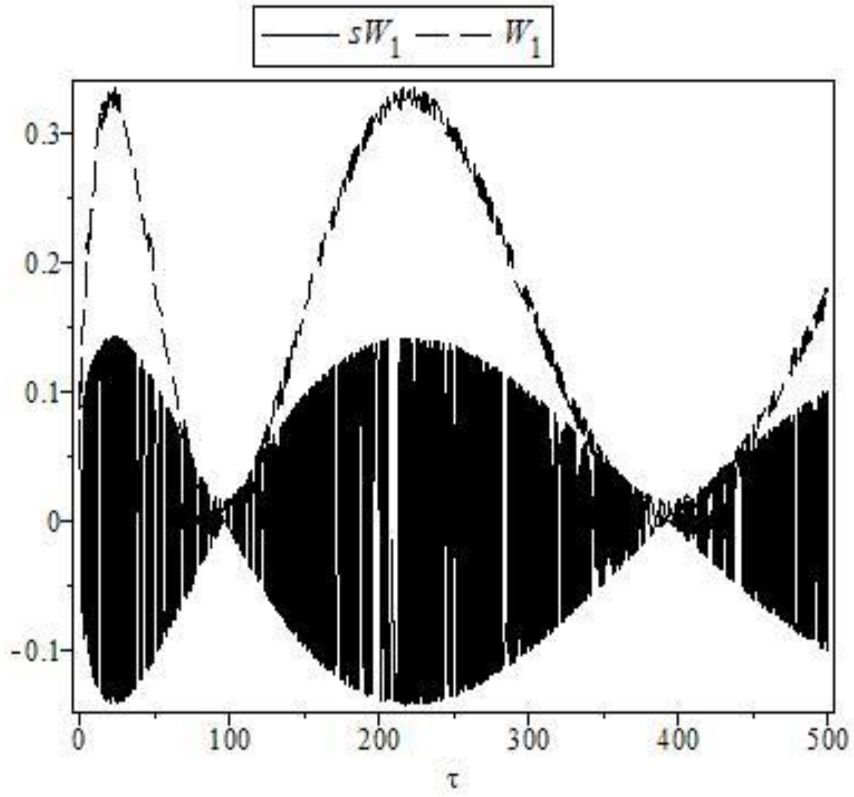


Figure 4.17 The energy W_1 and surplus of energy sW_1 for the source function $f_1(\xi,\tau)$ when $\alpha=-1$ ($0 \le \tau \le 500$), ξ is fixed.

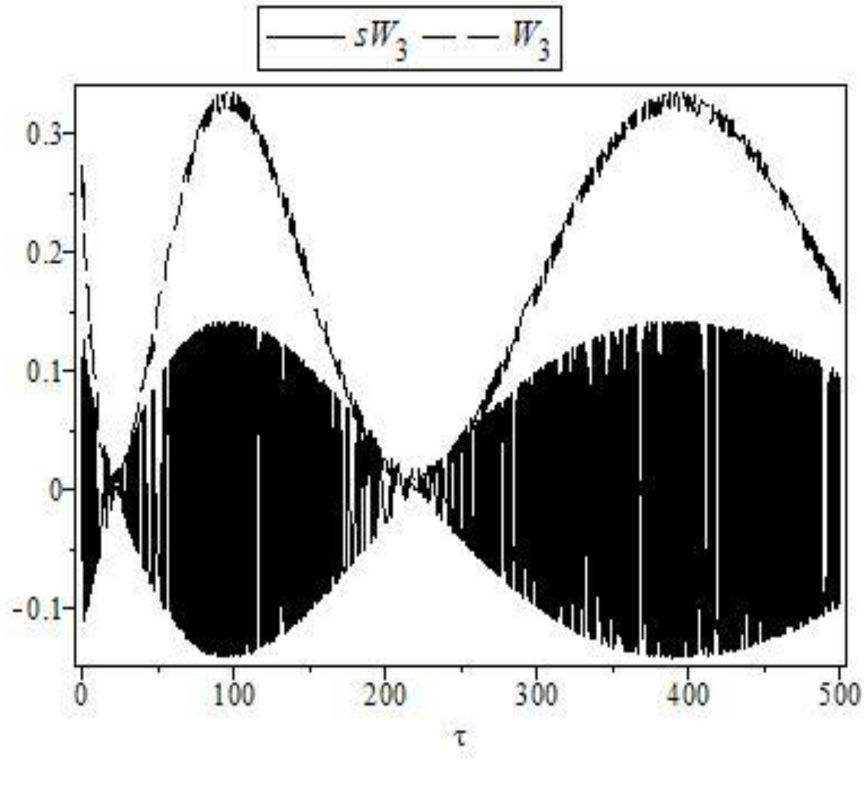


Figure 4.18 The energy W_3 and surplus of energy sW_3 for the source function $f_3(\xi,\tau)$ when $\alpha=-1$ ($0 \le \tau \le 500$), ξ is fixed.

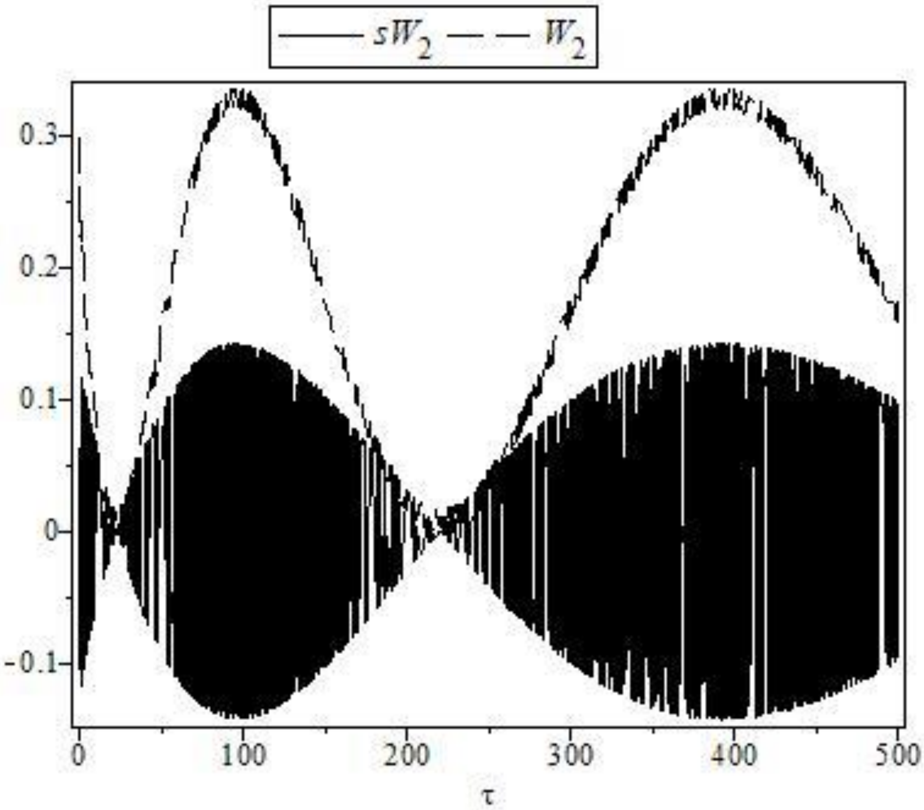


Figure 4.19 The energy W_2 and surplus of energy sW_2 for the source function $f_2(\xi, \tau)$ when $\alpha = -1$ $(0 \leq \tau \leq 500)$, ξ is fixed.

CHAPTER 5

RESULTS AND DISCUSSION

Those used in the study and the results obtained can be summarized to highlight the followings. In this dissertation, time-dependent strength of the electromagnetic fields produced by a source function in perfect electrically conductive surface waveguide have been discussed. It has been proposed an alternative technique for the study of electromagnetic field propagation in a hollow waveguide accompanied with transient signal transferring processes in the time-domain. This method is called the Evolutionary Approach to Electromagnetics (EAE). The study is designed in accordance with the theories of scientific ideas, theorists and engineers to present the waveguide theory in the time space.

The study of the time-domain modes has been shaped into two autonomous problems. The first one is the modal basis problem and the second one is the modal amplitude problem. The modal basis is common for time-harmonic and time-space modes. Therefore, the results obtained for the time-harmonic field modal basis in theory can be used freely in the time-space studies. On the other hand, the time-space amplitude studies need to be resolved the Klein-Gordon equation. Here, the symmetry properties of the Klein-Gordon equation have been used in terms of Group Theory [14]. The solution to the Klein-Gordon equation can be presented as a product of two functions if and only if the functions are specified in pairs in compliance with the properties of symmetry. A complete set of possible pairs is listed in Miller's list. Here, surplus of energy is invastigated in details via Airy functions by using the fifth pair of the list. This approach seems to be a promising way for solving other problems in time-space as long as it is obeyed the requirements of the time-dependent arguments and special functions of mathematical physics.

A brief summary of this study can be given as follwing. First, the fields have been taken from the Maxwell's equation systems with time derivative, ∂_t.Then the potentials have been obtained from Helmholtz equations by solving the two dimensional Dirichlet and Neumann boundary eigenvalue problems for the transverse Laplacian. The eigenvalues and the eigenfunctions related to these eigenvalues have been found from Helmholtz equations. These eigenvalues are real because the Helmholtz operator is a self-adjoint operator. The normalization of these eigenfunctions have been done by calculating the constant coefficients of the eigenfunctions under the boundary conditions. On the other hand, the series solutions of the time-dependent modal amplitudes have been derived from the Klein-Gordon equation. Therefore, it has been reached the fields by substitutions of amplitudes and potentials into the constitutive equations.The Klein-Gordon and Helmholtz equations are independent from each other in terms of the modal amplitude and the modal base. However, these two independent equations provide the evolution equations to be derived. In the discussion above, the Klein-Gordon equation has been solved accordance with the causality principle and satisfies the initial conditions, whereas the Helmholtz equation satisfies the boundary conditions. Moreover, both equations are invariant under Lorentz transformations. Finally, the energy and surplus of the energy have been taken into consideration. Formula of energetic wave process of stored energy in longitudinal and transversal field extensions and surplus of the energy have been shown. Some numerical examples have been given graphically.

In conclusion, the time-domain waveguide modes have been expressed analytically by a method of Evolutionary Approach to Electromagnetics (EAE). A time-dependent source function has been considered in a waveguide with perfect electric conductor surface. Specially, surplus of energy has been investigated in details via Airy functions. Thus, the energetic wave process of exchange by energy stored in the longitudinal and transverse field components has been introduced in the time-domain. Then surplus of energy, which occurs as a result of propagation of the signal function obtained analytically, has been discussed and some numerical examples have been shown graphically. Studies in transferring of signal analysis with the help of special functions have been made more recently [44]. In the future, the other possible solutions proposed from the Miller's eleven cases can be considered for the solution of different problems such as partially filled lossless and lossy waveguides.

REFERENCES

[1] Erden, F., (2009). Kavitelerde Zorlanmis Salinimlarin Zaman Domeni Analizlerinin Incelenmesi, Ph. D. Thesis, Gebze Institute of Technology, Electronics Engineering Department, Kocaeli.

[2] Ergin, S. and Sari, M., (2012). ‘‘Effects of the 4MAT instruction method on the achievement of high school students with different learning styles on the subjects of work, power and energy in physics’’, Energ. Educ. Sci. Tech-B, 2012, 4: 807–820.

[3] Yumrutas, R. and Unsal, M., (2012). ‘‘Modeling and performance analysis of a house heating system with ground coupled heat pump’’, Energ. Educ. Sci. Tech-A., 28:669–682.

[4] Taflove, A. and Hagness, S., (2005). Computational electrodynamics: the finite-difference time-domain method. Artech House, Boston.

[5] Gabriel, G., J., (1980). ‘‘Theory of electromagnetic transmission structures, Part I. Relativistic foundation and network formalism’’, Proc IEEE, 68:354–5366.

[6] Tretyakov, O., A., (1993). Essentials of nonstationary and nonlinear electromagnetic field theory, in analytical and numerical methods in electromagnetic wave theory. Hashimoto M, Idemen M, Tretyakov OA, Eds., ch. 3. Science House Co. Ltd., Tokyo.

[7] Aksoy, S. and Tretyakov, O.,(2002). ‘‘A Study of a time variant cavity system’’, J. Electromag Waves Applicat, 16:15535–1553.

[8] Eroglu, E., Aksoy S. and Tretyakov O. A., (2012), ‘‘Surplus of energy for time-domain waveguide modes’’, Energ Educ Sci Tech-A., 28, 1157–1164.

[9] Tretyakov, OA. and Akgün, O.,(2010). ‘‘Derivation of Klein-Gordon equation from Maxwell’s equations and study of relativistic time-domain waveguide modes.’’, Progress Electromagnetics Res (PIER), 105, 171–191.

[10] Eroglu, E., (2012). Dalga Kilavuzları Boyunca Gecici Sinyallerin Transferi, Ph.D. Thesis, Gebze Institute of Technology, Mathematics Department, Kocaeli.

[11] Aksoy, S. and Tretyakov, O., A., (2004). ‘‘Evolution equations for analytical study of digital signals in waveguides’’, J. Electromag Waves Appl, 17:263–270.

[12] Aksoy, S.and Antyufeyeva, M., (2005). ‘‘Time-domain cavity oscillations supported by a temporally dispersive dielectric’’, IEEE Transact Microwave Theory Techniq, 53:2465–2471.

[13] Eroğlu, E.and Ak, N., (2012). ‘‘Special functions in transferring of energy, a specific case: Airy function’’, Energ Educ Sci Tech-A. 2012, 30:719–726.

[14] Miller, WJr.,(1977). Symmetry and Separation of Variables, MA: Addison-Wesley Publication Co., Boston.

[15] Tretyakov, O.A., (1986). The modal basis method, Radiotekhnika in Electronica, 6, 1071-1082, Russia.

[16] Tretyakov, O.A., (1989). Waveguide evolutionary equations, Radiotekhnika in Electronica, 5, 917-926, Russia.

[17] Tretyakov, O.A., (1994). ‘‘Evolutionary equations for the theory of waveguides’’, IEEE AP-S Int. Symp. Dig., 1973-1976, Seattle, WA.

[18] Slater, J.C., (1946). ‘‘Microwave electronics’’, Rev. Mod. Physics, 18, 441-512.

[19] Kurokawa, K., (1958). ‘‘The Expansions of Electromagnetic Fields in Cavities’’, Microwave Theory and techniques, IRE Transactions on 6, 178-187.

[20] Van, Bladel, J. G., (1964). Electromagnetic Fields, McGraw-Hill, New York.

[21] Hillion, P., (1993). ‘‘Electromagnetism in a moving chiral medium’’, Phys. Rev. E, 48, 3060.

[22] Harmuth, H.F., (1994). Propagation of Electromagnetic Signals, MGM Hussian- Wold Scientific.

[23] Vallee, O. and Soares, M., (2004). Airy Functions and Applications to Physics, Imperial College Press, London.

[24] Abramowitz, M. and Stegun, I., (1965). Handbook of Mathematical Functions , Dover Publications, New York.

[25] Crandal, R.E., (1996). ‘‘On the quantum zeta function’’, J.Phys. A: Math. Gen. 29, 6795-6816.

[26] Miller, J.C.P., (1946). The Airy Integral, British Assoc. Adv.Sci. Mathematical Tables, Part-B, Cambridge Universty Press, London.

[27] Olver, F.W.J., (1974). Asymptotics and special functions, Academic Press, New York.

[28] Capson, E.T.,(1963).‘‘On the asymptotic expansion of Airy’s integral ’’, Proc.Glasgow Math. Assoc. 6, 113-115.

[29] Reid, W.H., (1995). ‘‘Integral representations for products of Airy functions’’, ZAMP 46, 159-170.

[30] Gordon, R.G., (1969). ‘‘New method for constructing wavefunctions for bound states and scattering’’, J. Chem. Phys. 51, 14-25.

[31] Schulten, Z.and Anderson D.G.M., (1979). ‘‘An algorithm for the evaluation of the complex Airy functions’’, J.Comput. Phys. 31, 60-75.

[32] Gradshetyn, I.S. and Ryzhik I.M., (1965). Tables of integrals, series and products, Academic Press, New York.

[33] Copson, E.T., (1967). Asymptotic expansions, Cambridge Universty Press.

[34] Olver, F.W.J., (1954). ‘‘The asymptotic solution of linear differential equations of the second order for large values of a parameter’’, Phil. Trans. R. Soc. London A 247, 307-327.

[35] Olver, F.W.J., (1954). ‘‘The asymptotic expansion of Bessel functions of large order’’, Phil. Trans. R. Soc. London A 247, 328-368.

[36] Chester, C.and Friedman, B., (1957). ‘‘An extension of the method of steepest descents’’, Proc. Camp.Philos. Soc. 53, 599-611.

[37] Jeffreys, H., (1942). Asymptotic solutions of linear differential equations, Phil. Mag.33, 451-456.

[38] Lebedev, N.N., (1965). Special Functions and Their Applications, Prentice-Hall INC, America.

[39] Fuchs, J.J.G., (1984). ‘‘The Theory of Differential Equations’’, Bull. Amer. Math.Soc. (N.S.) 10 (1), 1-26.

[40] Erratum, J.J.G., (1985). ‘‘Fuchs and The Theory of Differential Equations’’, Bull. Amer. Math.Soc.(N.S.) 12 (1), 182.

[41] Kragh, H., (1984). Equation with the many fathers. The Klein-Gordon equation in 1926, Am.J., Phys., 52,. (11), 1024-1033.

[42] Polyanin, A. D., (2002). Handbook of Linear Partial Differential Equations For Engineers and Scientists, Chapman &Hall/CRC Press, Boca Raton, FL.

[43] Hansen, T.B. and Yaghjian, A.D., (1999). Plane-wave theory of time-domain fields: Near-field scanning applications, Wiley-IEEE Press.

[44] Akgün, Ö. (2011). Real-Valued Time-Domain Modes in Hollow Waveguides, Ph. D.Thesis, Gebze Institute of Technology, Electronics Engineering Department, Gebze.

[45] Işık, Ö., (2014). Özel Fonksiyonlar Yardımıyla Sinyal Analizi, Ph. D.Thesis, Muğla Sıtkı Koçman University, Institute of Science, Mathematics Department, Muğla.

[46] Spiegel, M.R. (1971). Theory and Problems of Advanced Mathematics for Engineers and Scientists, McGraw-Hill Book Company, Singapore.

[47] Weyl, H. (1940). ‘‘The Method of Orthogonal Projection in Potential Theory’’, Duke Math. Journal, 7, 411-444.

APPENDIX-A

COMPLETE LIST OF SUBSTITUTIONS

A complete list of substitutions for $u(\xi,\tau)$ and $v(\xi,\tau)$ that factorize the solution to (4.18) as $f(u,v) = U(u).V(v)$ is given below. It is called the systems of coordinates (u,v) for the method of separation of variables in Klein-Gordon equation. After that list, some maple codes are given for some of the graphics above.

1. $\tau = u$ and $\xi = v$, where $-\infty < u < \infty, -\infty < v < \infty$ yield $f(u,v)$ as a product of exponential functions.

2. $\tau = u \cosh v$ *and* $\xi = u \sinh v$, where $0 \leq u < \infty, -\infty < v < \infty$ yield $f(u,v)$ as a product of an exponential and the Bessel functions.

3. $\tau = (u^2 + v^2)/2$ and $\xi = uv$, where $0 \leq u < \infty$, $-\infty < v < \infty$ yield $f(u,v)$ as a product of parabolic cylinder functions.

4. $\tau = uv$ and $\xi = (u^2 + v^2)/2$, where $0 \leq u < \infty$, $-\infty < v < \infty$ yield $f(u,v)$ as a product of parabolic cylinder functions.

5. $\tau + \xi = 2(u + v)$ and $\tau - \xi = (u - v)^2$, where $-\infty < u, v < \infty$ yield $f(u,v)$ as a product of Airy functions.

6. $\tau + \xi = \cosh[(u - v)/2]$ *and* $\tau - \xi = \sinh[(u + v)/2]$, where $-\infty < u, v < \infty$ yield $f(u,v)$ as a product of Mathieu functions.

7. $\tau + \xi = 2\sinh(u - v)$ *and* $\tau - \xi = \exp(u + v)$, where $-\infty < u, v < \infty$ yield $f(u,v)$ as a product of Bessel functions.

8. $\tau + \xi = 2\cosh(u - v)$ and $\tau - \xi = \exp(u + v)$, where $-\infty < u, v < \infty$ yield $f(u,v)$ as a product of Bessel functions.

9. $\tau = \sinh u \cosh v$ and $\xi = \cosh u \sinh v$, where $-\infty < u, v < \infty$ yield $f(u,v)$ as a product of Mathieu functions.

10. $\tau = \cosh u \cosh v$ and $\xi = \sinh u \sinh v$, where $-\infty < u < \infty, 0 \leq v < \infty$ yield $f(u,v)$ as a product of Mathieu functions.

11. $\tau = \cos u \cos v$ and $\xi = \sin u \sin v$,where$0 < u < 2\pi, 0 \leq v < \pi$ yield $f(u, v)$ as a product of Mathieu functions.

The substitutions given above specify orthogonal systems of coordinates (u, v). Besides, there are non-orthogonal systems which enable us to separate the variables u and v in the KGE, see [14].

APPENDIX-B

SOME MAPLE CODES

```
>##f12:Ai(u)*Bi(v) AIRY FUNCTION


>restart;

alpha:=-1;


y1:=-0.2;

y2:=0.6;

tau1:=0;

tau2:=15;

u:=(4^(1/3))*(alpha-(tau+xi)/4+(((tau-xi)^(1/2))/2));

v:=(4^(1/3))*(alpha-(tau+xi)/4-(((tau-xi)^(1/2))/2));


f2:=AiryAi((4^(1/3))*(alpha-(tau+xi)/4+(((tau-
xi)^(1/2))/2)))*AiryBi((4^(1/3))*(alpha-(tau+xi)/4-(((tau-
xi)^(1/2))/2)));


ttau:=diff(-f2,tau);

txi:=diff(f2,xi);


N:=1500;          #N=numpoints
```

```
th:=2;           #th is thickness
h2:=piecewise(0>tau,0,f2); #Causality

xi:=tau-0.05;

with(plots):

a2:=plot(h2,tau=tau1..tau2,numpoints=N,thickness=th,view=[t
au1..tau2,y1..y2],axes=boxed,legend=typeset(F[2]),color=bla
ck,symbol=box,linestyle=solid);

t3:=plot(ttau,tau=tau1..tau2,numpoints=N,thickness=th,view=
[tau1..tau2,y1..y2],axes=boxed,legend=typeset(A[2]),color=b
lack,symbol=box,linestyle=dot);

t4:=plot(txi,tau=tau1..tau2,numpoints=N,thickness=th,view=[
tau1..tau2,y1..y2],axes=boxed,legend=typeset(B[2]),color=bl
ack,symbol=box,linestyle=dash);

display({a2,t3,t4});

>##SURPLUS of ENERGY

>restart;
alpha:=-5;

delta:=0.05:
```

```
tau1:=0:

tau2:=15:

th:=2:

N:=1500:

u:=(4^(1/3))*(alpha-(tau+xi)/4+(((tau-xi)^(1/2))/2));

v:=(4^(1/3))*(alpha-(tau+xi)/4-(((tau-xi)^(1/2))/2));


f4:=AiryBi(u)*AiryBi(v):

ttau:=diff(-f4,tau):

txi:=diff(f4,xi):

xi:=tau-delta:


W:=plot(((ttau^2)+(txi^2)+(f4^2))/2,tau=tau1..tau2,thicknes
s=th,numpoints=N,color=black,linestyle=dash,axes=boxed):


sW:=plot(((ttau^2)-
(txi^2))/2,tau=tau1..tau2,thickness=th,numpoints=N,color=bl
ack,axes=boxed):


with(plots):


t1:=textplot([1,0.29,"W"],align={above,
right},font=[times,bold,11]):


t2:=textplot([13.3,0.165,"sW"],align={above,
right},font=[times,bold,12]):


display({W,sW,t1,t2})
```

CURRICULUM VITAE

PERSONAL INFORMATION

Name Surname : Nevra EREN

Date of birth and place :01/01/1973 - AĞRI

Foreign Languages : English

E-mail :nevraboul@mynet.com

EDUCATION

Degree	Department	University	Date of Graduation
M. Sc.	Science and Literature	Marmara University	2009
B. Sc.	Science and Literature	Bosphorus University	1999
Kadıköy Kenan Evren High School, Mathematics Department			1991

WORK EXPERIENCE

Year	Corporation / Institute
2015	Üsküdar Yavuz Selim İ.H.Secondary School, Teacher
2010-2014	Üsküdar Henza Akın Çolakoğlu Anadolu İ.H.High School, Teacher
2009-2010	Ümraniye Eczacı Neşem High School, Teacher
2007-2009	Ümraniye Dudullu 75. Yıl Cumhuriyet High School, Teacher
2006	Batman Yaylıca Primary School, Teacher

PUBLISHMENTS

Papers

Eroğlu E., Ak N., Köklü K., Özdemir O.Z., Çelik N., Eren N., (2012), ‘’Special functions in transferring of energy; a specific case: ‘’ Airy function ’’, Eneg Educ Sci Tech-A 2012;30:719–726.

Conference Papers

Köklü, K., Eroğlu, E., Eren, N., (2012). ‘‘Transferring of energy via Special functions, a specific case: ‘‘Airy function’’, 1st International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2012), Proceeding Book, 120.